Energy Method

mi@seu.edu.cn

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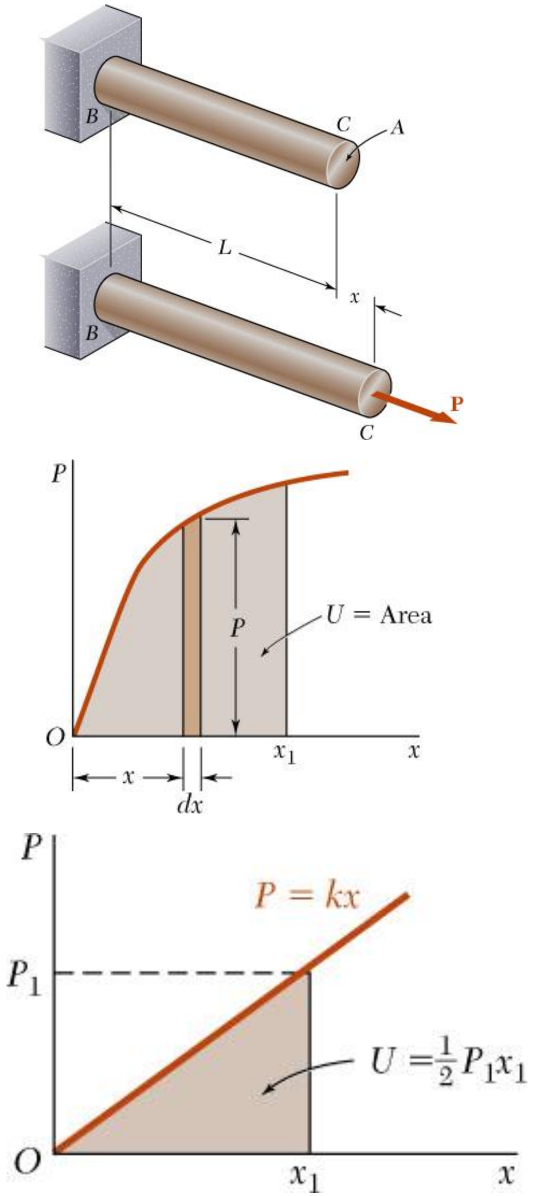
Work done by External Loads and Strain Energy



- A uniform rod is subjected to a slowly increasing load

- The *elementary work* done by the load *P* as the rod elongates by a small d*x* is

$$dW = P\,dx = elementary\ work$$

which is equal to the area of width d*x* under the load-deformation diagram.

- The *total work* done by the load for a deformation x_1,

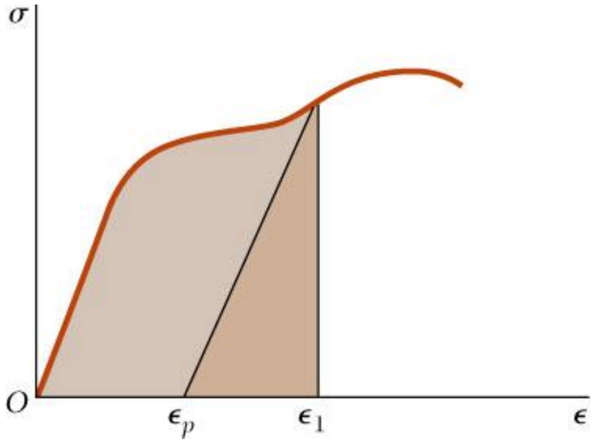
$$W = \int_0^{x_1} P\,dx = total\ work = strain\ energy$$

which results in an increase of *strain energy* in the rod.

- In the case of a linear elastic deformation,

$$W = \int_0^{x_1} P\,dx = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}P_1x_1 = U$$

Strain Energy Density



- To eliminate the effects of size, evaluate the strain-energy per unit volume,

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A} \frac{dx}{L}$$

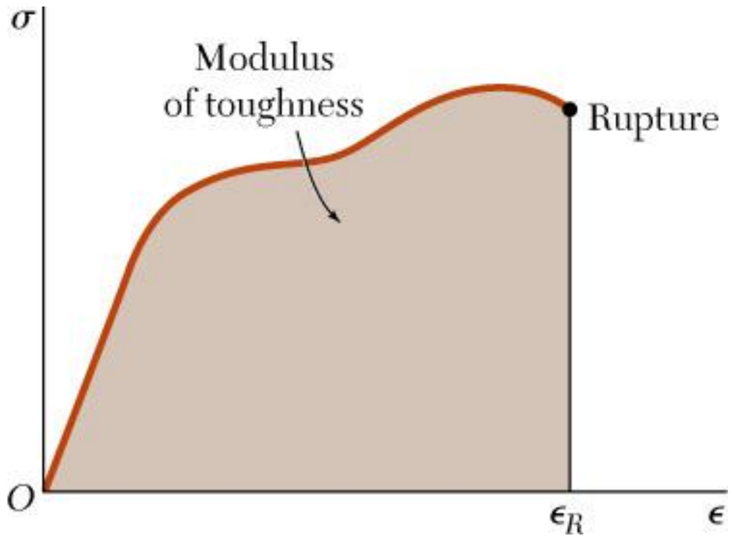
$$u = \int_0^{\varepsilon_1} \sigma_x \, d\varepsilon_x = strain\,energy\ density$$

- The total strain energy density resulting from the deformation is equal to the area under the curve to ε_1.

- As the material is unloaded, the stress returns to zero but there is a permanent deformation. Only the strain energy represented by the triangular area is recovered.

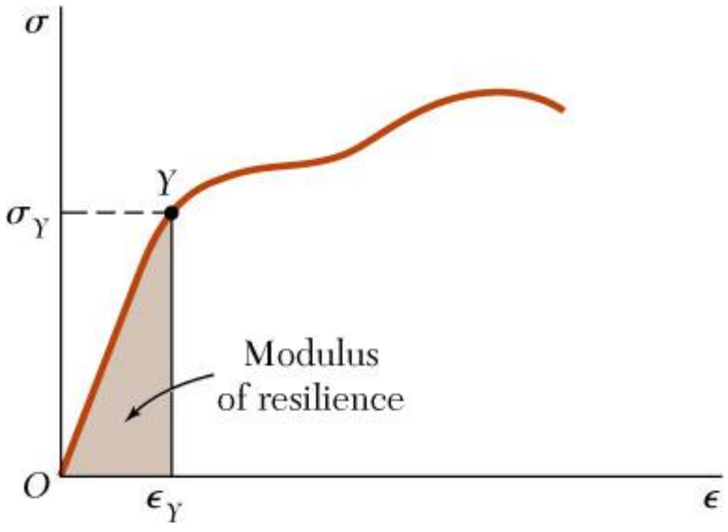
- Remainder of the energy spent in deforming the material is dissipated as heat.

Strain Energy Density



- The strain energy density resulting from setting $\varepsilon_1 = \varepsilon_R$ is the *modulus of toughness*.

- The energy per unit volume required to cause the material to rupture is related to its ductility as well as its ultimate strength.



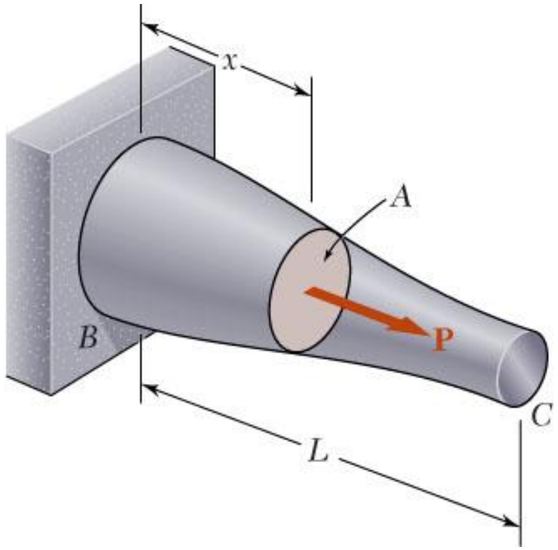
- If the stress remains within the proportional limit,

$$u = \int_0^{\varepsilon_1} E\varepsilon_x \, d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

- The strain energy density resulting from setting $\sigma_1 = \sigma_Y$ is the *modulus of resilience*.

$$u_Y = \frac{\sigma_Y^2}{2E} = \text{modulus of resilience}$$

Strain Energy due to Normal Stresses



- In an element with a nonuniform stress distribution,

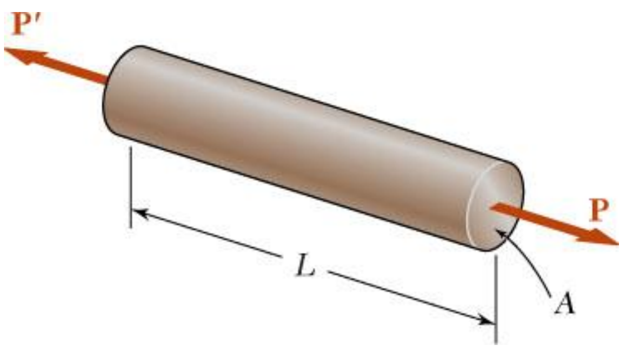
$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \qquad U = \int u \, dV = \text{total strain energy}$$

- For values of $u < u_Y$, i.e., below the proportional limit,

$$U = \int \frac{\sigma_x^2}{2E} dV = \textit{elastic strain energy}$$

- Under axial loading, $\sigma_x = P/A \qquad dV = A\,dx$

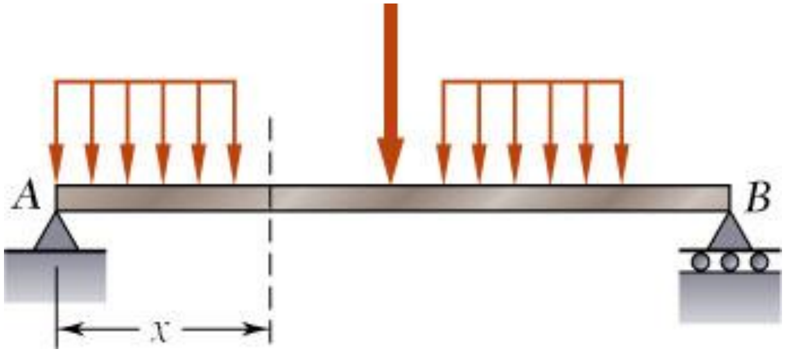
$$U = \int_0^L \frac{P^2}{2AE} dx$$



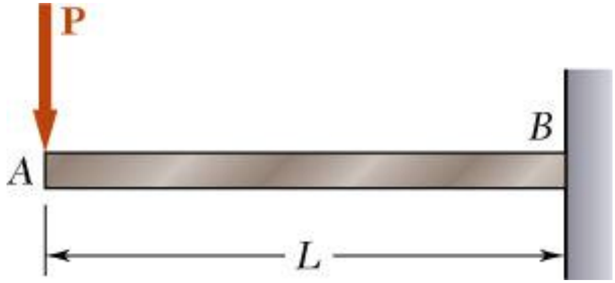
- For a rod of uniform cross-section,

$$U = \frac{P^2 L}{2AE}$$

Strain Energy due to Normal Stresses



$$\sigma_x = \frac{M\,y}{I}$$



- For a beam subjected to a bending load,

$$U = \int \frac{\sigma_x^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} dA\,dx = \int_0^L \frac{M^2}{2EI^2} \left(\int_A y^2 dA \right) dx$$

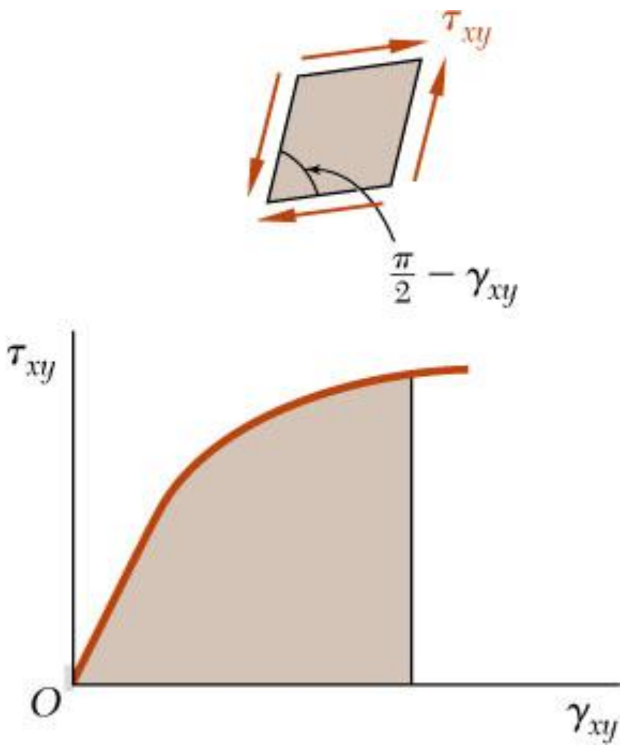
$$= \int_0^L \frac{M^2}{2EI} dx$$

- For an end-loaded cantilever beam,

$$M = -Px$$

$$U = \int_0^L \frac{P^2 x^2}{2EI} dx = \frac{P^2 L^3}{6EI}$$

Strain Energy due to Shearing Stresses



- For a material subjected to plane shearing stresses,

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy}$$

- For values of τ_{xy} within the proportional limit,

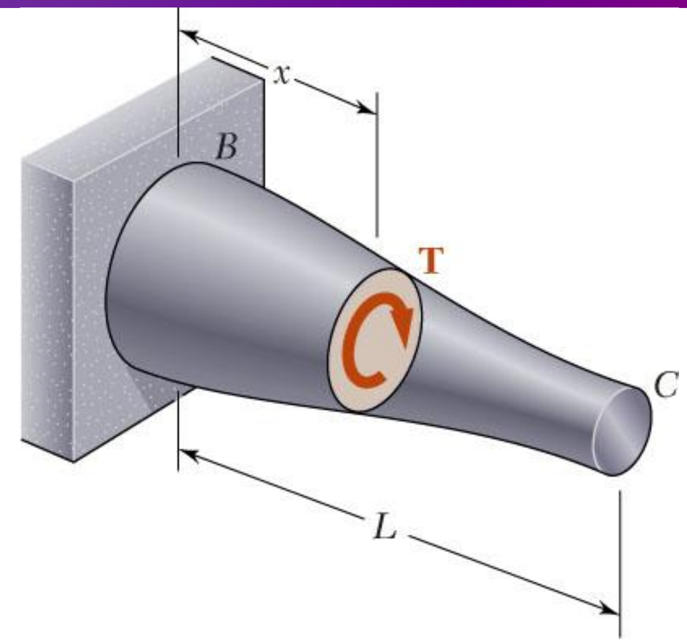
$$u = \tfrac{1}{2} G\gamma_{xy}^2 = \tfrac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G}$$

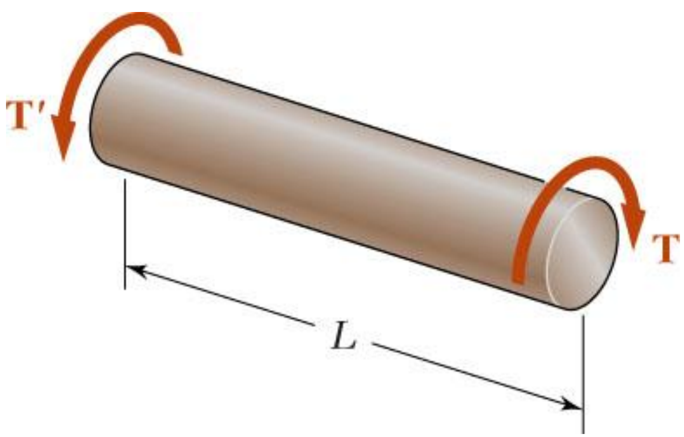
- The total strain energy is found from

$$U = \int u\, dV$$

$$= \int \frac{\tau_{xy}^2}{2G}\, dV$$

Strain Energy due to Shearing Stresses





- For a shaft subjected to a torsional load,

$$U = \int \frac{\tau_{x\theta}^2}{2G} dV = \int \frac{T^2 \rho^2}{2GI_p^2} dV$$

- Setting $dV = dA\,dx$,

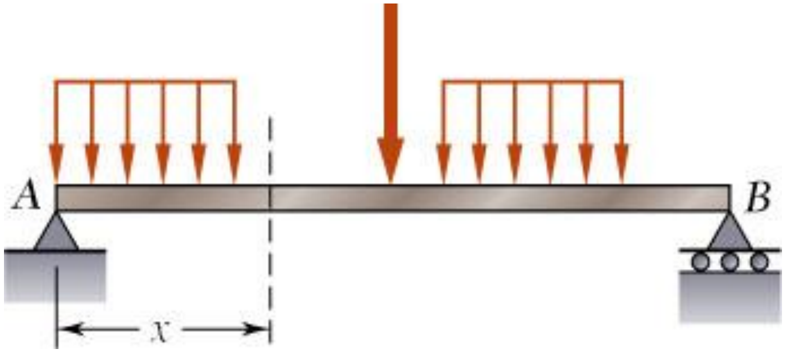
$$U = \int_0^L \int_A \frac{T^2 \rho^2}{2GI_p^2} dA\,dx = \int_0^L \frac{T^2}{2GI_p^2} \left(\int_A \rho^2 dA \right) dx$$

$$= \int_0^L \frac{T^2}{2GI_p} dx$$

- In the case of a uniform shaft,

$$U = \frac{T^2 L}{2GI_p}$$

Strain Energy due to Shearing Stresses



$$\tau_{xy} = \frac{F_S S_z^*}{I_z b}$$

- Define the *form factor* for shear

$$f_S = \frac{A}{I_z^2}\int_A \left(\frac{S_z^*}{b}\right)^2 dA$$

- Dimensionless
- Unique for each specific cross-sectional area

- For a beam subjected to a bending load,

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{1}{2G}\left(\frac{F_S S_z^*}{I_z b}\right)^2 \mathrm{d}V$$

- Setting $dV = dA\,dx$,

$$U = \int_0^L \int_A \frac{1}{2G}\left(\frac{F_S S_z^*}{I_z b}\right)^2 \mathrm{d}A\mathrm{d}x$$

$$= \int_0^L \frac{F_S^2}{2GI_z^2}\int_A \left(\frac{S_z^*}{b}\right)^2 \mathrm{d}A\mathrm{d}x$$

- In terms of the *form factor*

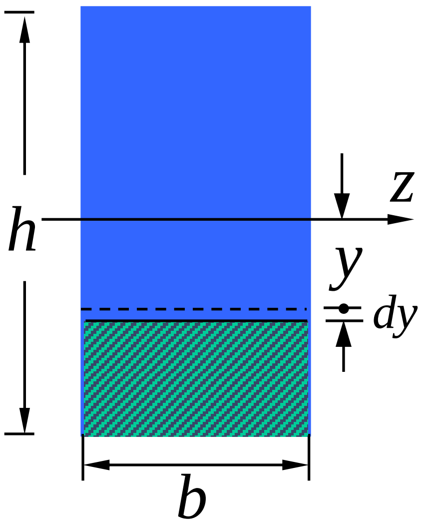
$$U = \int_0^L f_S \frac{F_S^2}{2GA}\mathrm{d}x$$

Form Factors for Shear

- The *form factor* for rectangular cross-section

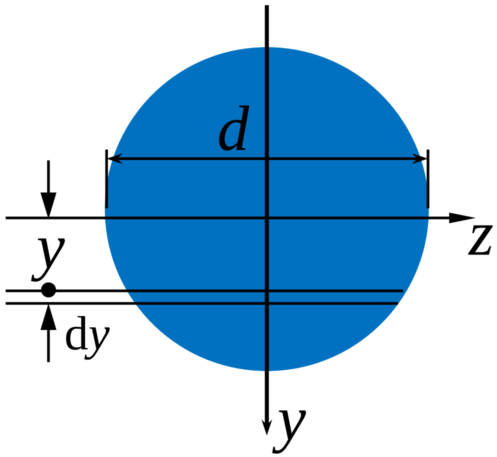
$$f_S = \frac{A}{I_z^2}\left[\int_A \left(\frac{S_z^*}{b}\right)^2 \mathrm{d}A\right]$$

$$= \frac{(bh)}{\left(bh^3/12\right)^2}\left[b\int_{-h/2}^{h/2}\left(\frac{1}{b}\frac{b}{2}\left(\frac{h^2}{4}-y^2\right)\right)^2 \mathrm{d}y\right] = 1.20$$



- The *form factor* for circular cross-section

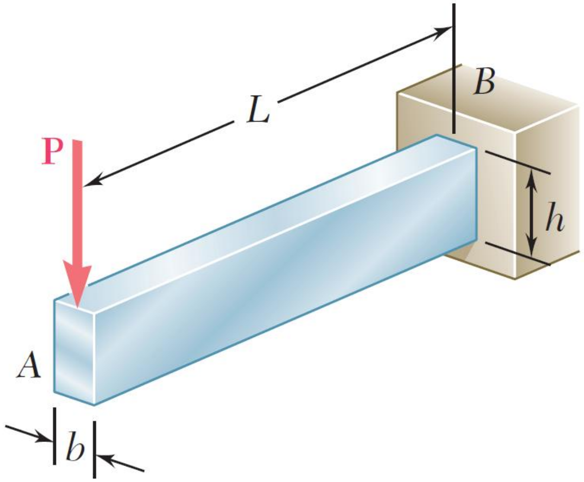
$$f_S = \frac{\left(\pi d^2/4\right)}{\left(\pi d^4/64\right)^2}\int_{-d/2}^{d/2}\left[\int_{-\sqrt{d^2/4-y'^2}}^{\sqrt{d^2/4-y'^2}}\left(\frac{\frac{2}{3}\left(d^2/4-y^2\right)^{3/2}}{2\left(d^2/4-y^2\right)^{1/2}}\right)^2 \mathrm{d}z\right]\mathrm{d}y \approx 1.08$$



- For thin-walled circular tubes: $f_S = 1.95$.
- For thin-walled square tubes: $f_S = 2.35$.

Strain Energy due to Bending & Transverse Shear

- For an end-loaded cantilever beam
- The total strain energy due to both bending and transverse shear



$$U = U_b + U_s = \int_0^L \frac{P^2 x^2}{2EI} dx + \int_0^L \frac{6}{5} \frac{P^2}{2GA} \mathrm{d}x = \frac{P^2 L^3}{6EI} + \frac{3P^2 L}{5GA}$$

$$= \frac{P^2 L^3}{6EI}\left[1 + \frac{18EI}{5GAL^2}\right] = \frac{P^2 L^3}{6EI}\left[1 + \frac{3Eh^2}{10GL^2}\right]$$

- For steel, take $E/G \approx 2.6$
  - For $h/L = 1/5$: $\frac{3Eh^2}{10GL^2} = 0.0312$
  - For $h/L = 1/6$: $\frac{3Eh^2}{10GL^2} = 0.0217$
  - For $h/L = 1/8$: $\frac{3Eh^2}{10GL^2} = 0.0122$
  - For $h/L = 1/10$: $\frac{3Eh^2}{10GL^2} = 0.0078$
- **The strain energy due to transverse shear is of importance only in the case of very short deep beams, i.e., for large *h*/*L* ratios.**

Strain Energy due to a General State of Stress

- Previously found strain energy due to uniaxial stress and plane shearing stress. For a general state of stress,

$$u = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

- With respect to the principal axes for an elastic, isotropic body,

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right]$$

$$= u_v + u_d$$

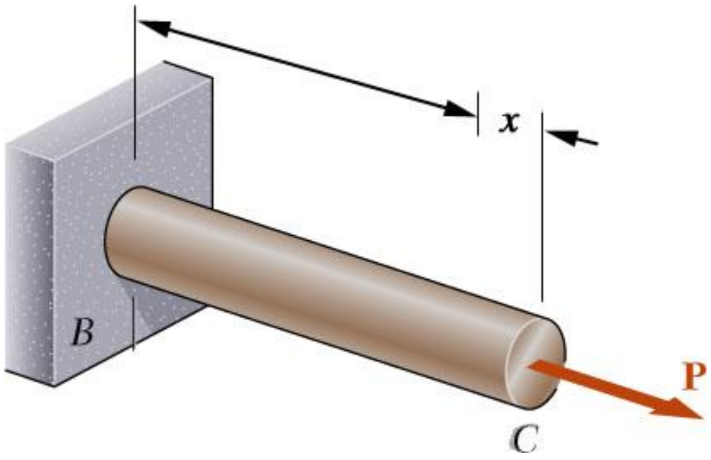
$$u_v = \frac{1-2\nu}{6E}\left(\sigma_a + \sigma_b + \sigma_c\right)^2 = \text{due to volume change}$$

$$u_d = \frac{1}{12G}\left[\left(\sigma_a - \sigma_b\right)^2 + \left(\sigma_b - \sigma_c\right)^2 + \left(\sigma_c - \sigma_a\right)^2\right] = \text{due to distortion}$$

- Basis for the *maximum distortion energy* failure criteria,

$$u_d < \left(u_d\right)_Y = \frac{\sigma_Y^2}{6G} \quad \text{for a tensile test specimen}$$

Work and Energy under a Single Load



- Previously, we found the strain energy by integrating the energy density over the volume.
  For a uniform rod,

$$U = \int u\, dV = \int \frac{\sigma^2}{2E} dV$$

$$= \int_0^L \frac{\left(P_1/A\right)^2}{2E} A dx = \frac{P_1^2 L}{2EA}$$

- Strain energy may also be found from the work of the single load P_1,

$$W = \int_0^{x_1} P\, dx$$

- For an elastic deformation,

$$W = \int_0^{x_1} P\, dx = \int_0^{x_1} kx\, dx = \tfrac{1}{2} k x_1^2 = \tfrac{1}{2} P_1 x_1$$

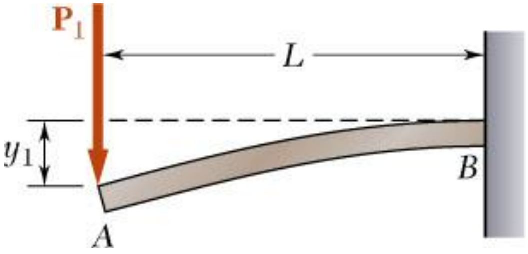
- Knowing the equivalence between strain energy and work,

$$U = W \quad \Rightarrow x_1 = \frac{W}{P_1/2} = \frac{P_1 L}{AE}$$

Work and Energy under a Single Load

- Strain energy may be found from the work of other types of single concentrated loads.

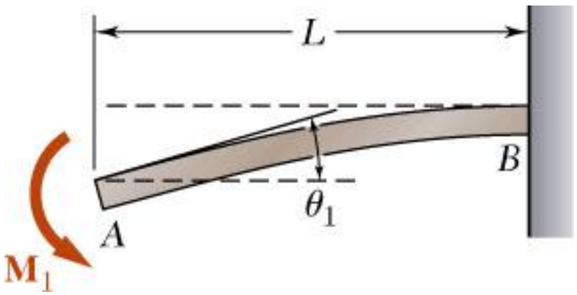
- Transverse load



$$W = \int_0^{y_1} P\,dy = \tfrac{1}{2} P_1 y_1$$

$$U = \tfrac{1}{2} P_1 \left(\frac{P_1 L^3}{3EI} \right) = \frac{P_1^2 L^3}{6EI}$$

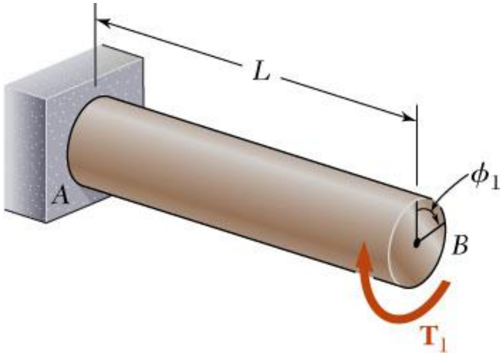
- Bending Moment



$$W = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2} M_1 \theta_1$$

$$U = \tfrac{1}{2} M_1 \left(\frac{M_1 L}{EI} \right) = \frac{M_1^2 L}{2EI}$$

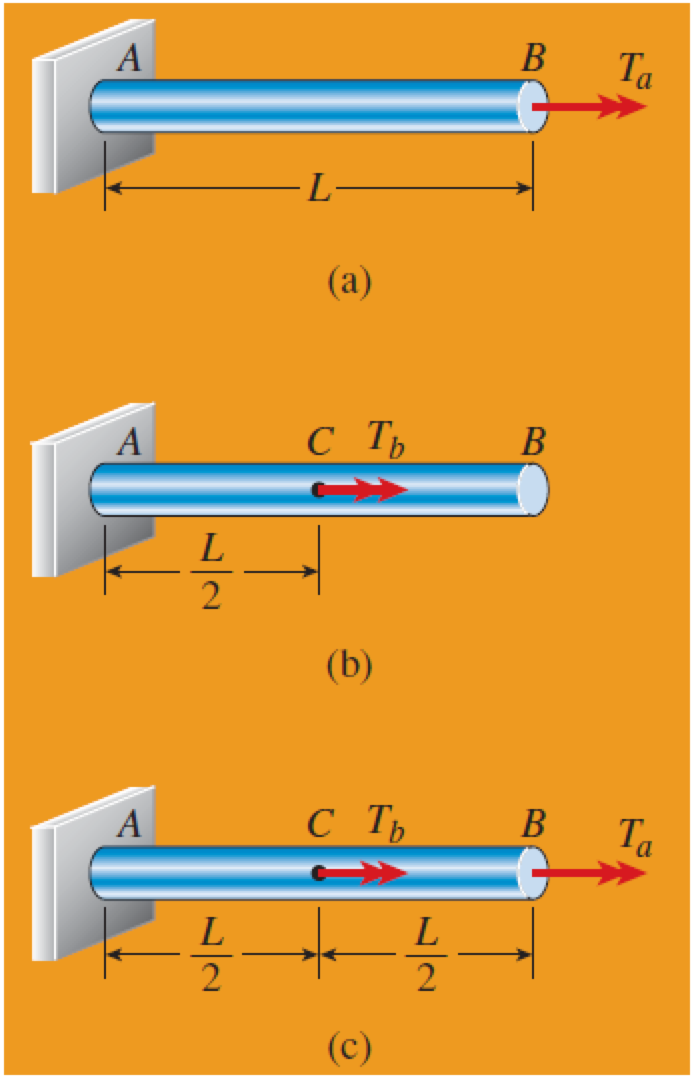
- Twisting Moment



$$W = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2} T_1 \phi_1$$

$$U = \tfrac{1}{2} T_1 \left(\frac{T_1 L}{G I_p} \right) = \frac{T_1^2 L}{2 G I_p}$$

Strain Energy cannot be Superposed



- A solid circular bar is fixed at one end and free at the other. Three different loading conditions are to be considered. For each case of loading, obtain a formula for the strain energy stored in the bar.

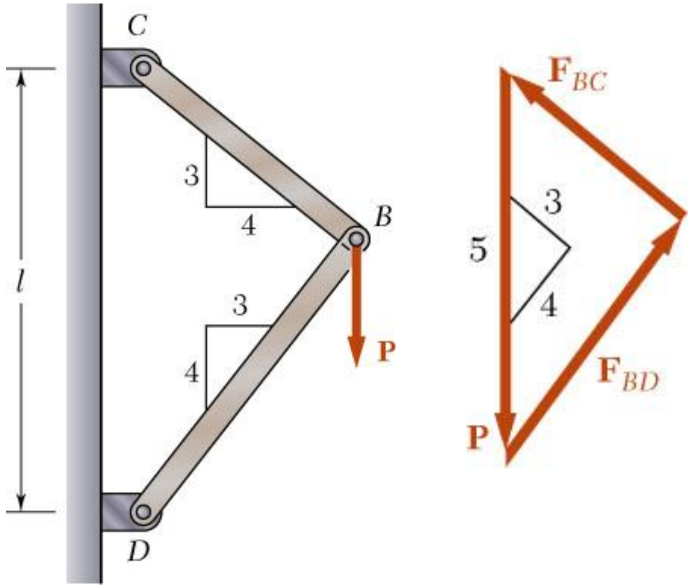
$$U_a = \frac{T_a^2 L}{2GI_p}$$

$$U_b = \frac{T_b^2 (L/2)}{2GI_p} = \frac{T_b^2 L}{4GI_p}$$

$$U_c = \frac{T_a^2 (L/2)}{2GI_p} + \frac{(T_a + T_b)^2 (L/2)}{2GI_p} = \frac{T_a^2 L}{2GI_p} + \frac{T_a T_b L}{2GI_p} + \frac{T_b^2 L}{4GI_p}$$

- The strain energy produced by the two loads acting simultaneously is not equal to the sum of the strain energies produced by the loads acting separately.
- Strain energy is a quadratic function of the loads, not a linear function.

Sample Problem



From the given geometry,

$$L_{BC} = 0.6l \quad L_{BD} = 0.8l$$

From statics,

$$F_{BC} = +0.6P \quad F_{BD} = -0.8P$$

- If the strain energy of a structure due to a single concentrated load is known, then the equality between the work of the load and energy may be used to find the deflection.

- Strain energy of the structure,

$$U = \frac{F_{BC}^2 L_{BC}}{2AE} + \frac{F_{BD}^2 L_{BD}}{2AE}$$

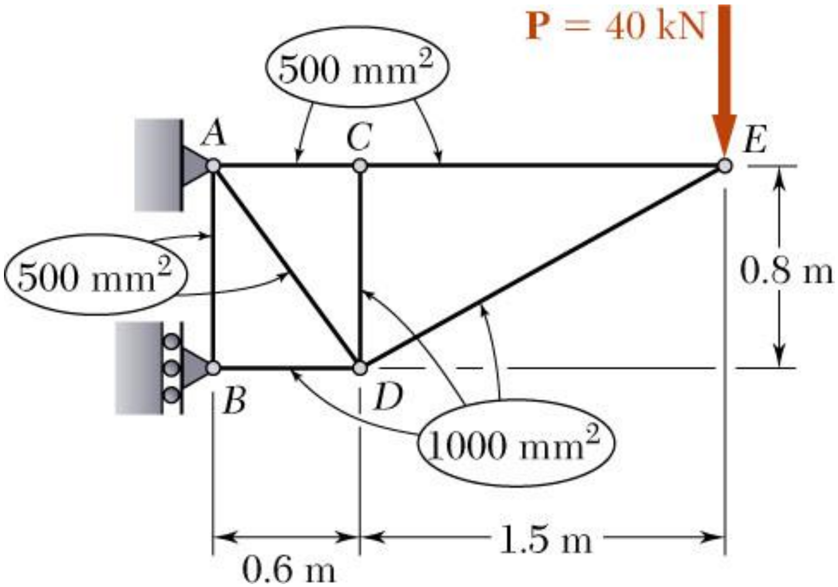
$$= \frac{P^2 l\left[(0.6)^3 + (0.8)^3\right]}{2AE} = 0.364\frac{P^2 l}{AE}$$

- Equating work and strain energy,

$$U = 0.364\frac{P^2 L}{AE} = \tfrac{1}{2} P y_B$$

$$y_B = 0.728\frac{Pl}{AE}$$

Sample Problem

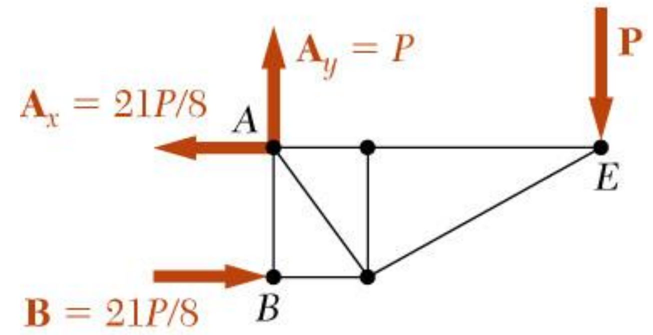


Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using E = 73 GPa, determine the vertical deflection of the point E caused by the load P.

Solution:

- Find the reactions at A and B from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member.
- Evaluate the strain energy of the truss due to the load P.
- Equate the strain energy to the work of P and solve for the displacement.

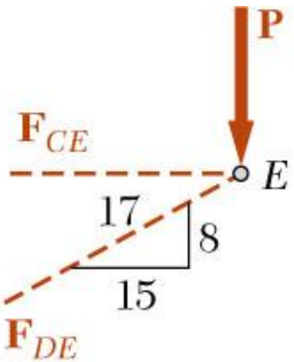
Solution:

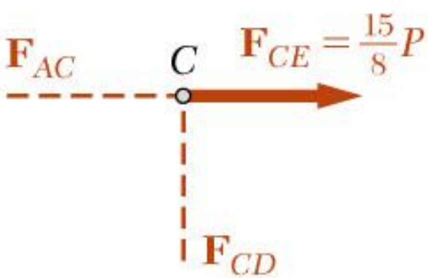


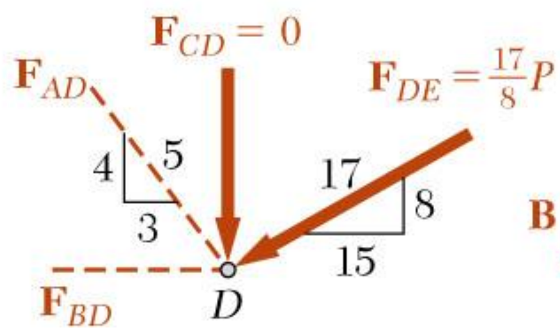
- Find the reactions at A and B from a free-body diagram of the entire truss.

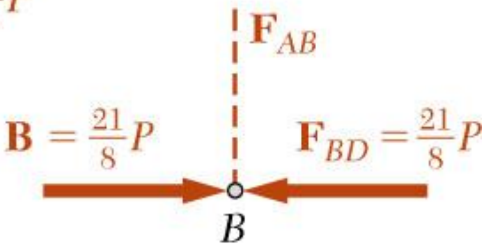
  $A_x = -21P/8 \qquad A_y = P \qquad B = 21P/8$

- Apply the method of joints to determine the axial force in each member.









$F_{DE} = -\frac{17}{8}P$

$F_{CE} = +\frac{15}{8}P$

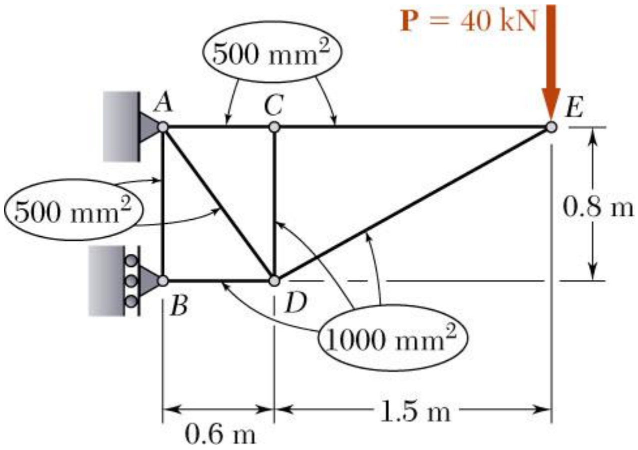
$F_{AC} = +\frac{15}{8}P$

$F_{CD} = 0$

$F_{DE} = \frac{5}{4}P$

$F_{CE} = -\frac{21}{8}P$

$F_{AB} = 0$



Member	F_i	L_i, m	A_i, m²	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

- Evaluate the strain energy of the truss due to the load *P*.

$$U = \sum \frac{F_i^2 L_i}{2A_i E} = \frac{1}{2E} \sum \frac{F_i^2 L_i}{A_i}$$

$$= \frac{1}{2E}\left(29700P^2\right)$$

- Equate the strain energy to the work by *P* and solve for the displacement.

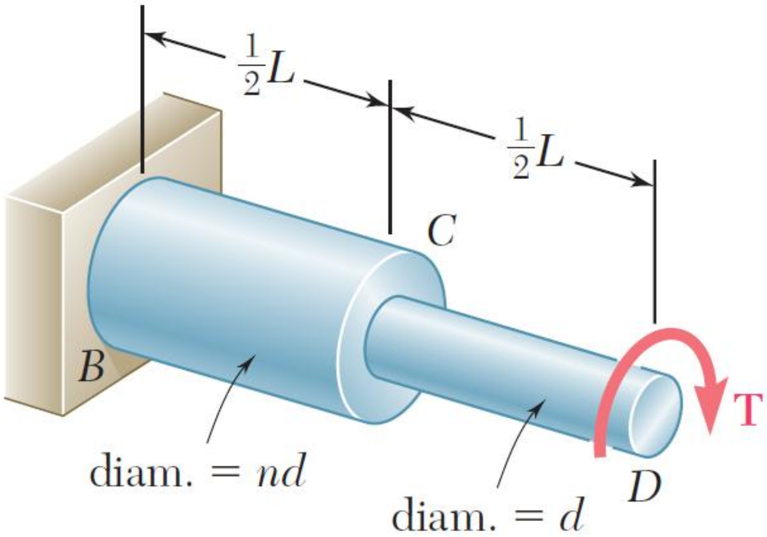
$$\tfrac{1}{2} P y_E = U$$

$$y_E = \frac{2U}{P} = \frac{2}{P}\left(\frac{29700P^2}{2E}\right)$$

$$y_E = \frac{\left(29.7\times10^3\right)\left(40\times10^3\right)}{73\times10^9}$$

$$y_E = 16.27\text{mm} \downarrow$$

Sample Problem



- Determine the angle of twist at end D of the shaft by equating the strain energy to the work done by the load.

- Solution

$$U_n = U_{DC} + U_{CB} = \frac{T^2 (L/2)}{2GI_p} + \frac{T^2 (L/2)}{2G(n^4 I_p)}$$

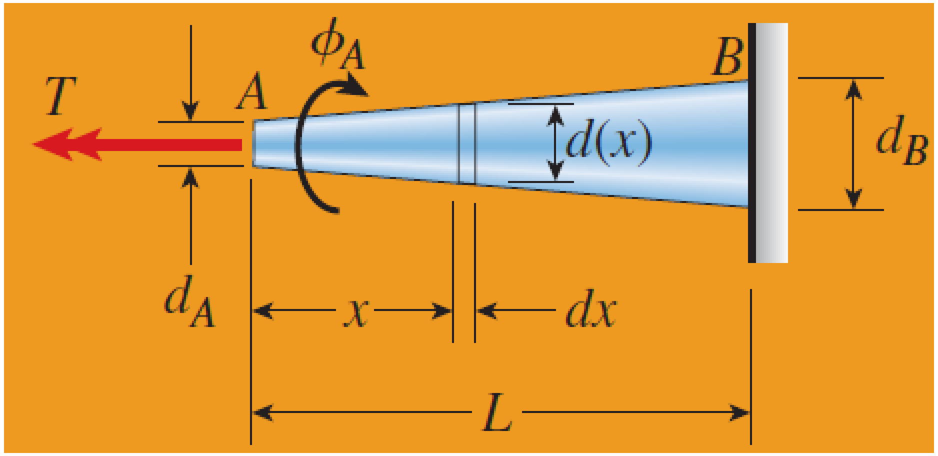
$$= \frac{1}{2}\left(1 + \frac{1}{n^4}\right)\frac{T^2 L}{2GI_p}$$

$$\phi_n = \frac{U_n}{T/2} = \frac{1}{2}\left(1 + \frac{1}{n^4}\right)\frac{TL}{GI_p}$$

$$\Rightarrow \begin{cases} U_1 = \dfrac{T^2 L}{2GI_p}, U_2 = \dfrac{17}{32}U_1, U_3 = \dfrac{41}{81}U_1, U_n = \dfrac{n^4+1}{2n^4}U_1 \\ \phi_1 = \dfrac{TL}{GI_p}, \phi_2 = \dfrac{17}{32}\phi_1, \phi_3 = \dfrac{41}{81}\phi_1, \phi_n = \dfrac{n^4+1}{2n^4}\phi_1 \end{cases}$$

- For a given allowable stress, increasing the diameter of portion *BC* of the shaft results in a decrease of the overall energy-absorbing capacity of the shaft.

Sample Problem



- Determine the angle of twist at end *A* of the shaft by equating the strain energy to the work done by the load.

- Solution

$$d_x = d_A + \frac{x}{L}\left(d_B - d_A\right)$$

$$I_{px} = \frac{\pi d_x^4}{32} = \frac{\pi}{32}\left(d_A + \frac{x}{L}\left(d_B - d_A\right)\right)^4$$

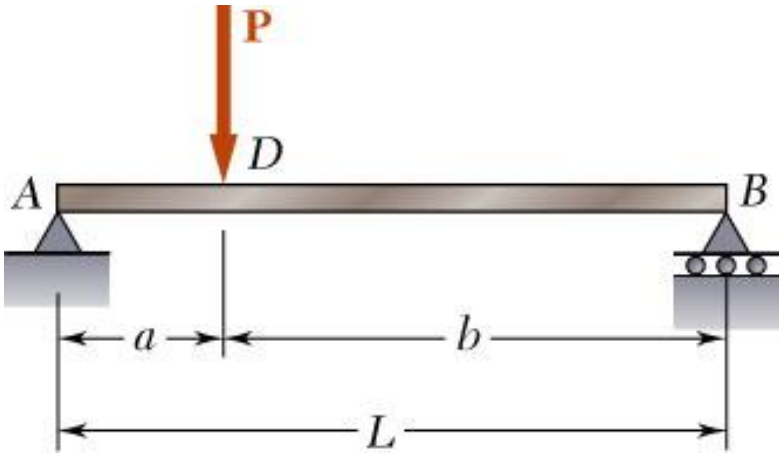
$$U = \int_0^L \frac{T^2 dx}{2GI_{px}}$$

$$= \int_0^L \frac{T^2 dx}{2G\dfrac{\pi}{32}\left(d_A + \dfrac{x}{L}\left(d_B - d_A\right)\right)^4}$$

$$= \frac{16T^2}{\pi G}\frac{L}{3\left(d_B - d_A\right)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

$$\phi_A = \frac{U}{T/2} = \frac{32TL}{3\pi G\left(d_B - d_A\right)}\left(\frac{1}{d_A^3} - \frac{1}{d_B^3}\right)$$

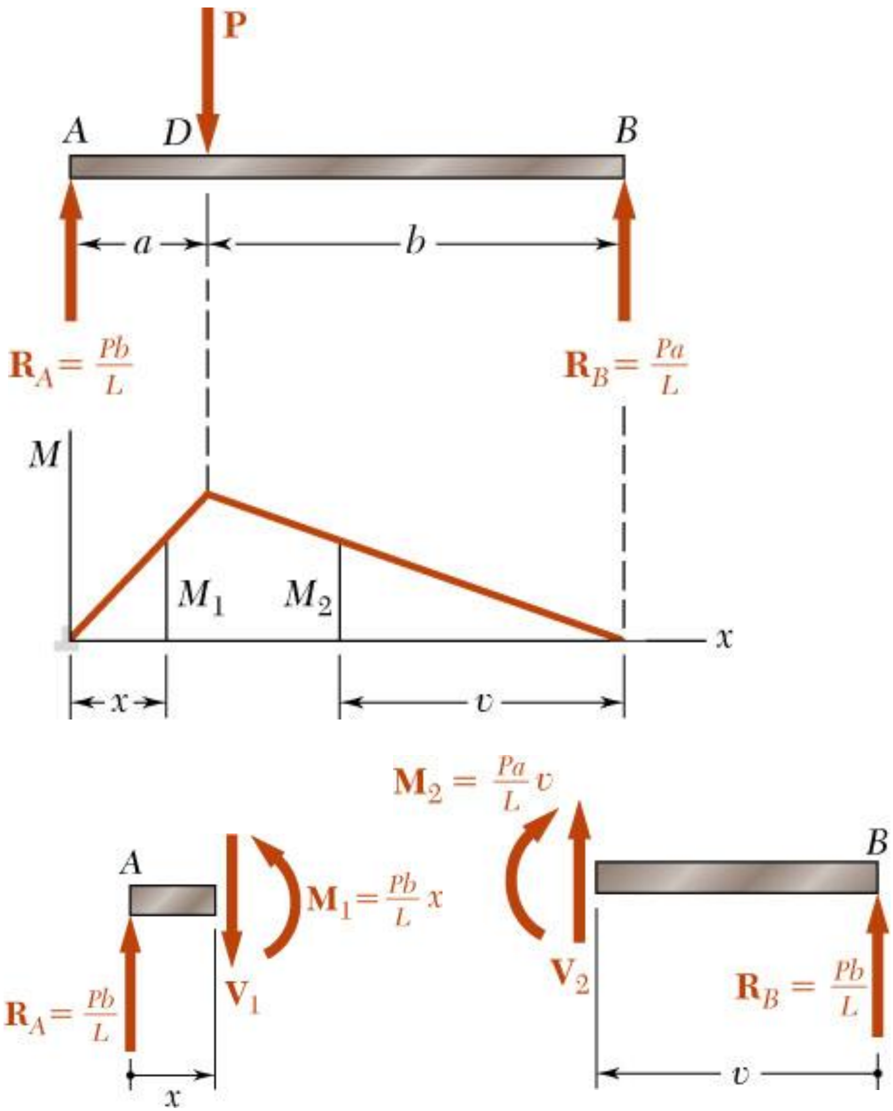
Sample Problem



Taking into account only the normal stresses due to bending, determine the vertical displacement at cross-section *D* of the beam for the loading shown.

Solution:

- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.
- Develop a diagram of the bending moment distribution.
- Integrate over the volume of the beam to find the strain energy.
- Find the vertical displacement at *D* by equating the work done by the transverse force to the strain energy.



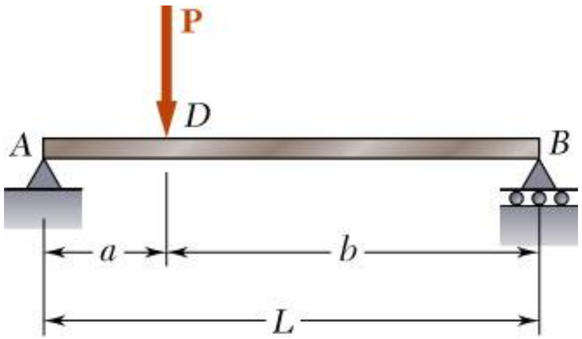
Solution:

- Determine the reactions at *A* and *B* from a free-body diagram of the complete beam.

$$R_A = \frac{Pb}{L} \qquad R_B = \frac{Pa}{L}$$

- Develop a diagram of the bending moment distribution.

$$M_1 = \frac{Pb}{L}x \qquad M_2 = \frac{Pa}{L}v$$



Over the portion *AD*,

$$M_1 = \frac{Pb}{L}x$$

Over the portion *BD*,

$$M_2 = \frac{Pa}{L}x$$

- Integrate over the volume of the beam to find the strain energy.

$$U = \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$

$$= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}x\right)^2 dx$$

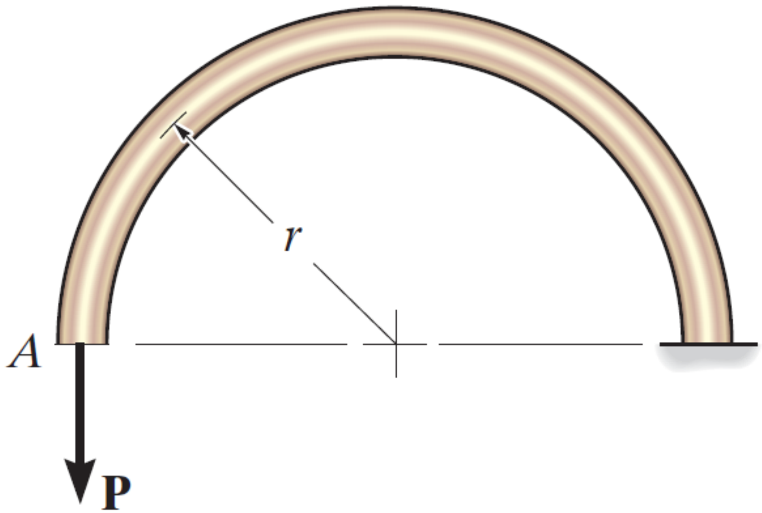
$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$U = \frac{P^2a^2b^2}{6EIL}$$

- Find the vertical displacement at *D* by equating the work done by the transverse force to the strain energy.

$$y_D = \frac{U}{P/2} = \frac{Pa^2b^2}{3EIL}$$

Sample Problem



- Determine the vertical displacement at *A*. Only consider the strain energy due to bending. Assume constant flexural rigidity *EI*.

- Solution

$$M = Pr\left(1-\cos\theta\right)$$

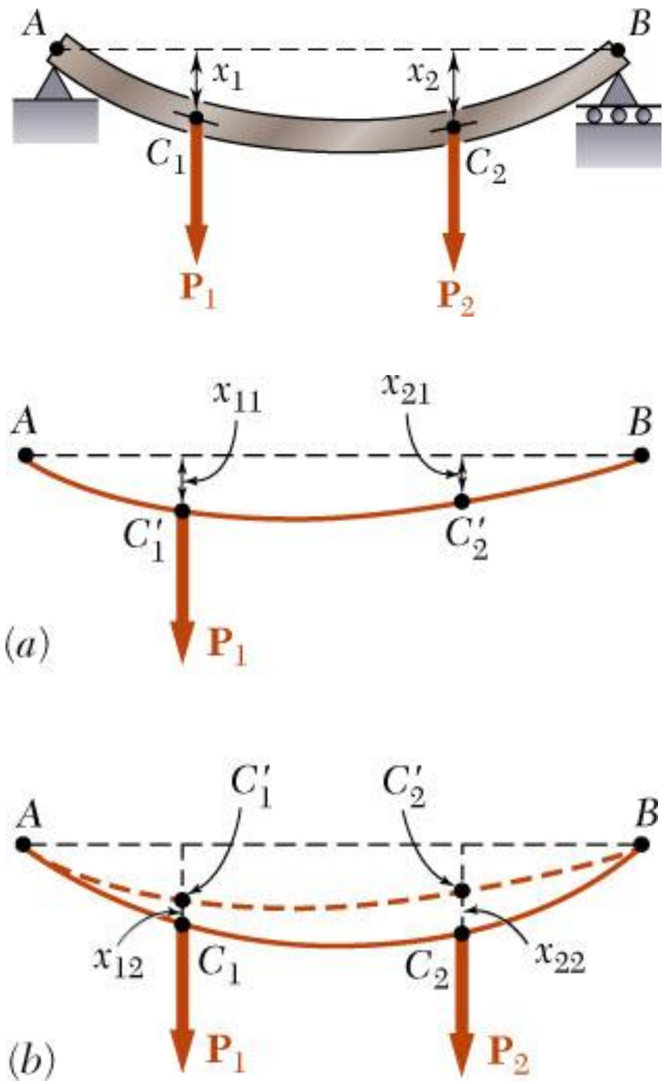
$$U = \int_0^L \frac{M^2 ds}{2EI} = \int_0^\pi \frac{\left(Pr\left(1-\cos\theta\right)\right)^2 r d\theta}{2EI}$$

$$= \frac{P^2 r^3}{2EI}\int_0^\pi \left(1-\cos\theta\right)^2 d\theta$$

$$= \frac{3\pi P^2 r^3}{4EI}$$

$$y_A = \frac{U}{P/2} = \frac{3\pi P r^3}{2EI}$$

Work and Energy under Several Loads



- Deflections of an elastic beam subjected to two concentrated loads,

$$x_1 = x_{11} + x_{12} = \alpha_{11}P_1 + \alpha_{12}P_2$$

$$x_2 = x_{21} + x_{22} = \alpha_{21}P_1 + \alpha_{22}P_2$$

- Compute the strain energy in the beam by evaluating the work done by slowly applying P_1 followed by P_2,

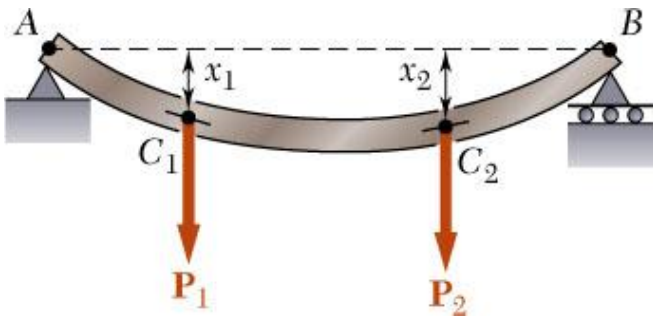
$$U = \tfrac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Reversing the application sequence yields

$$U = \tfrac{1}{2}\left(\alpha_{22}P_2^2 + 2\alpha_{21}P_2P_1 + \alpha_{11}P_1^2\right)$$

- Strain energy expressions must be equivalent. It follows that $\alpha_{12}=\alpha_{21}$ (*Maxwell's reciprocal theorem*).

Castigliano's Second Theorem



Carlo Alberto Castigliano (9 November 1847 – 25 October 1884) Italian mathematician and physicist.

- Strain energy for any elastic structure subjected to two concentrated loads,

$$U = \tfrac{1}{2}\left(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2\right)$$

- Differentiating with respect to the loads,

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1$$

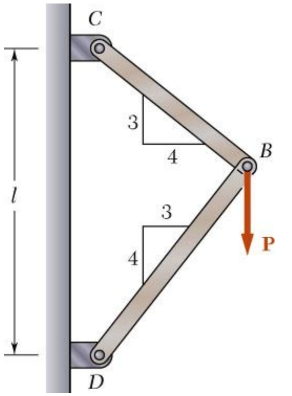
$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2$$

- ***Castigliano's theorem***: For an elastic structure subjected to *n* loads, the deflection y_j of the point of application of P_j can be expressed as

$$y_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \qquad \phi_j = \frac{\partial U}{\partial T_j}$$

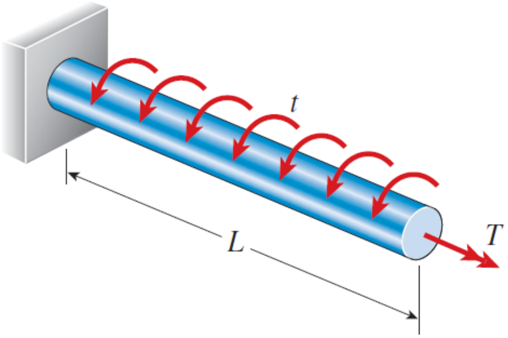
Castigliano's Second Theorem

- Castigliano's theorem is simplified if the differentiation w.r.t. the load is performed before the integration or summation.



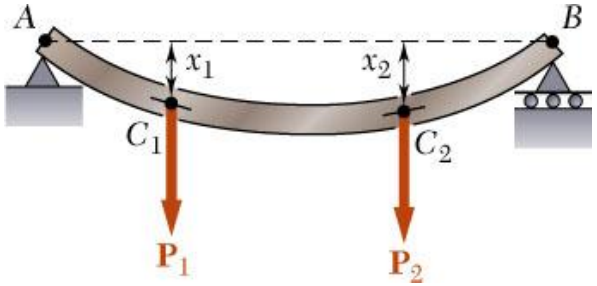
- For tension / compression

$$U = \sum_{i=1}^{n} \frac{F_i^{\,2} L_i}{2A_i E} \qquad y_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j}$$



- For torsion

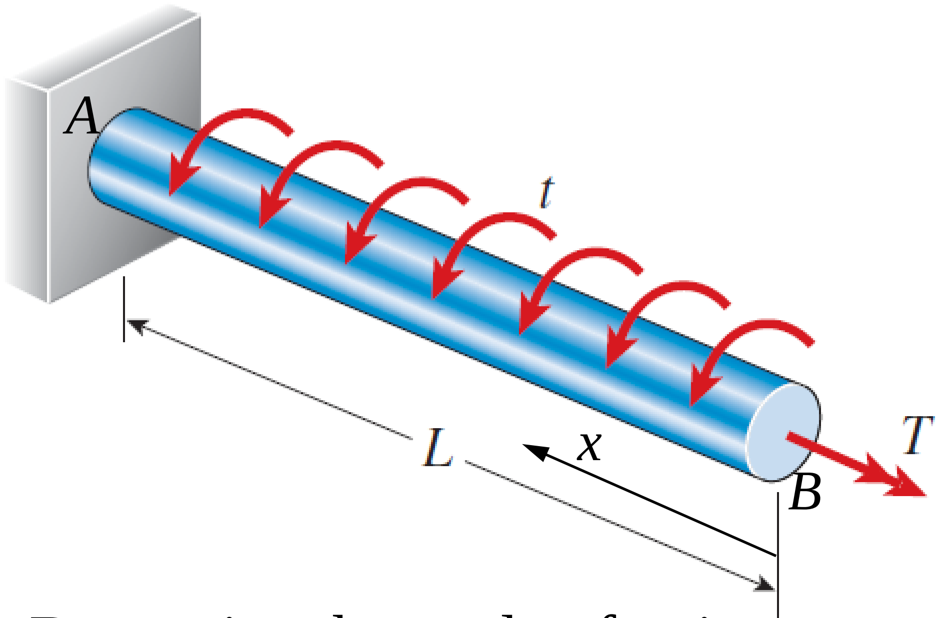
$$U = \int_0^L \frac{T^2}{2GI_p} dx, \quad \phi_j = \frac{\partial U}{\partial T_j} = \int_0^L \frac{T}{GI_p} \frac{\partial T}{\partial T_j} dx$$



- For bending

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad y_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P_j} dx$$

Sample Problem



- Determine the angle of twist at end *A* of the shaft.

- Solution

$$T_x = T + tx$$

$$U = \int_0^L \frac{T_x^2 dx}{2GI_p} = \int_0^L \frac{(T+tx)^2\, dx}{2GI_p}$$

$$= \frac{1}{2GI_p}\int_0^L \left(T^2 + 2Ttx + t^2x^2\right)dx$$

$$= \frac{T^2L + TtL^2 + t^2L^3/3}{2GI_p}$$

$$\phi_B = \frac{\partial U}{\partial T} = \frac{TL + tL^2/2}{GI_p}$$

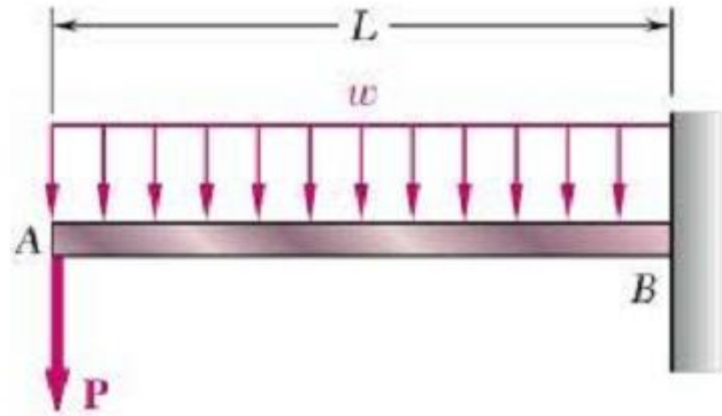
- Alternatively

$$\phi_B = \frac{\partial U}{\partial T} = \int_0^L \frac{T_x}{GI_p}\frac{\partial T_x}{\partial T}dx = \int_0^L \frac{(T+tx)}{GI_p}dx = \frac{TL + tL^2/2}{GI_p}$$

Sample Problem

- The cantilever beam *AB* supports a uniformly distributed load w and a concentrated load *P* as shown. Knowing that *L* = 2 m, *w* = 4 kN/m, *P* = 6 kN, and *EI* = 5 MNm^2, determine the deflection at *A*.

- Solution



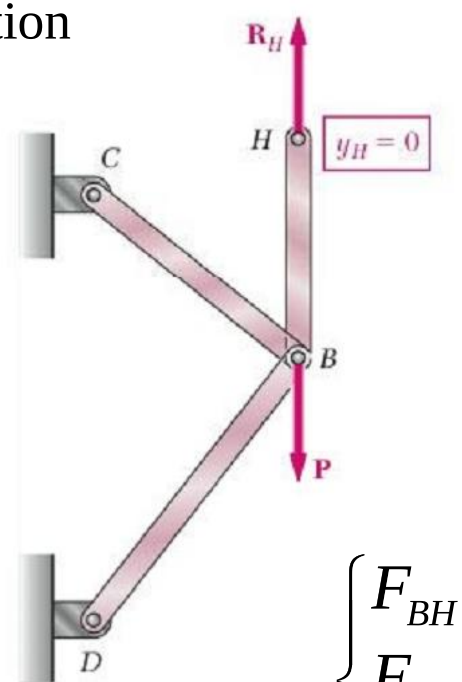
$$M = -\left(Px + \frac{1}{2}wx^2\right) \Rightarrow \frac{\partial M}{\partial P} = -x$$

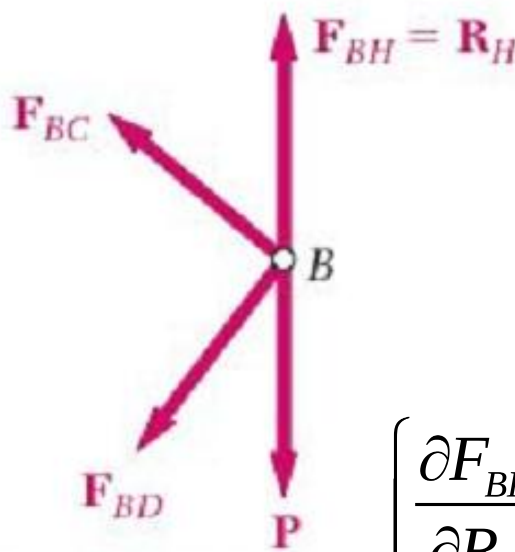
$$\Rightarrow y_A = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dx = \frac{1}{EI}\int_0^L \left(Px + \frac{1}{2}wx^2\right)xdx = \frac{1}{EI}\left(\frac{PL^3}{3} + \frac{wL^4}{8}\right)$$

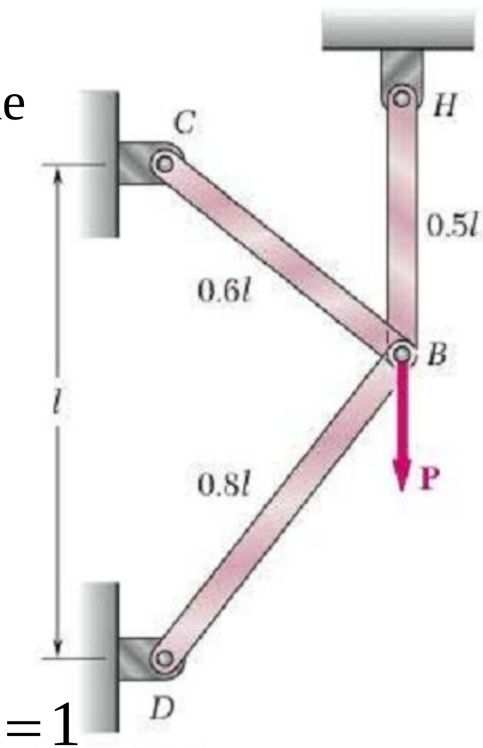
$$= \frac{1}{5\times10^6}\left[\frac{6\times10^3\times2^3}{3} + \frac{4\times10^3\times2^4}{8}\right] = 4.8mm \downarrow$$

Statically Indeterminate Truss

- A load P is supported at B by three rods of the same material and the same cross-sectional area A. Determine the axial force in each rod.
- Solution







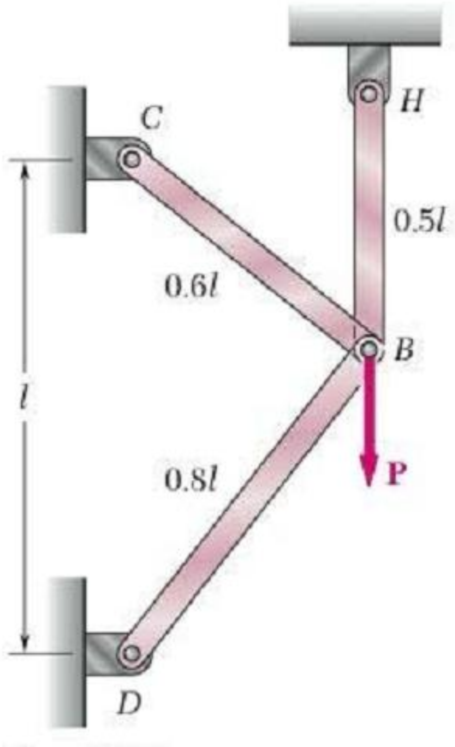
$$\begin{cases} F_{BH} = R_H \\ F_{BC} = 0.6P - 0.6R_H \\ F_{BD} = 0.8R_H - 0.8P \end{cases} \Rightarrow \begin{cases} \dfrac{\partial F_{BH}}{\partial R_H} = 1 \\ \dfrac{\partial F_{BC}}{\partial R_H} = -0.6 \\ \dfrac{\partial F_{BD}}{\partial R_H} = 0.8 \end{cases}$$

$$0 = y_H = \frac{\partial U}{\partial R_H} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial R_H}$$

$$= \frac{F_{BH} L_{BH}}{AE} \frac{\partial F_{BH}}{\partial R_H} + \frac{F_{BC} L_{BC}}{AE} \frac{\partial F_{BC}}{\partial R_H} + \frac{F_{BD} L_{BD}}{AE} \frac{\partial F_{BD}}{\partial R_H}$$

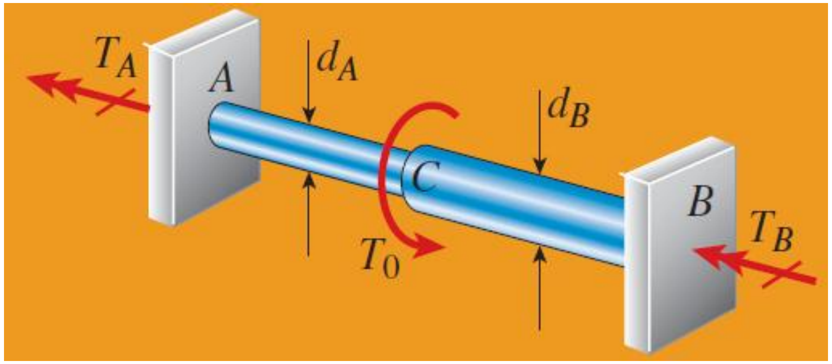
$$= \frac{1}{AE} \begin{bmatrix} (R_H)(0.5L)(1) \\ +(0.6P - 0.6R_H)(0.6L)(-0.6) \\ +(0.8R_H - 0.8P)(0.8L)(0.8) \end{bmatrix}$$

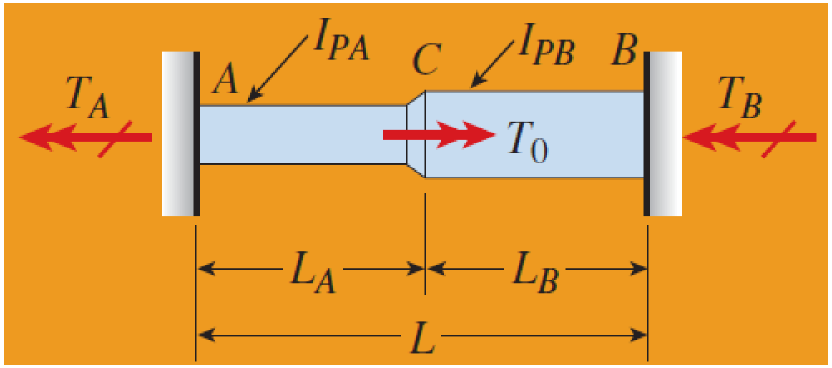
$$\Rightarrow R_H = 0.593P \Rightarrow \begin{cases} F_{BH} = 0.593P \\ F_{BC} = 0.244P \\ F_{BD} = -0.326P \end{cases}$$



Statically Indeterminate Shafts

- Determine (a) the reactive torques at the ends, (b) the angle of rotation at the cross section where the load T_0 is applied.





- Solution:

$$\phi_j = \frac{\partial U}{\partial T_j} = \sum_i \frac{T_i L_i}{GI_{pi}} \frac{\partial T_i}{\partial T_j} = \frac{T_A L_A}{GI_{pA}} \frac{\partial T_A}{\partial T_j} + \frac{T_B L_B}{GI_{pB}} \frac{\partial T_B}{\partial T_j}$$

$$0 = \phi_A = \frac{T_A L_A}{GI_{pA}} \frac{\partial T_A}{\partial T_A} + \frac{(T_0 - T_A) L_B}{GI_{pB}} \frac{\partial (T_0 - T_A)}{\partial T_A} = \frac{T_A L_A}{GI_{pA}} - \frac{(T_0 - T_A) L_B}{GI_{pB}}$$

$$0 = \phi_B = \frac{(T_0 - T_B) L_A}{GI_{pA}} \frac{\partial (T_0 - T_B)}{\partial T_B} + \frac{T_B L_B}{GI_{pB}} \frac{\partial T_B}{\partial T_B} = -\frac{(T_0 - T_B) L_A}{GI_{pA}} + \frac{T_B L_B}{GI_{pB}}$$

Statically Indeterminate Shafts

$$\Rightarrow T_A = \frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} T_0, \qquad T_B = \frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} T_0$$

$$\Rightarrow \phi_C = \frac{T_A L_A}{G I_{pA}} \frac{\partial T_A}{\partial T_0} + \frac{T_B L_B}{G I_{pB}} \frac{\partial T_B}{\partial T_0} = \frac{L_A L_B T_0}{G\left(L_B I_{pA} + L_A I_{pB}\right)} = \frac{T_A L_A}{G I_{pA}} = \frac{T_B L_B}{G I_{pB}}$$

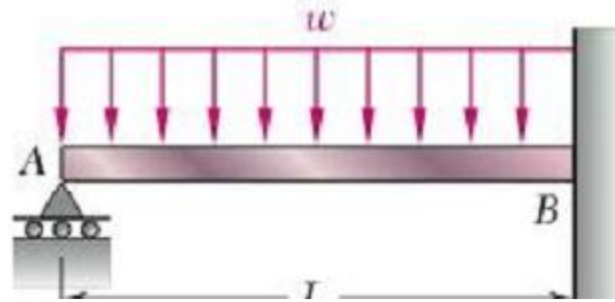
- For the special case of $d_A = d_B$:

$$\Rightarrow T_A = \frac{L_B}{L} T_0, \qquad T_B = \frac{L_A}{L} T_0$$

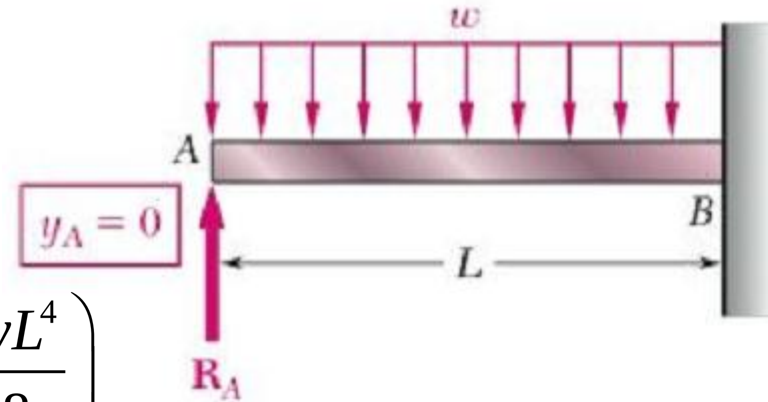
$$\Rightarrow \phi_C = \frac{L_A L_B T_0}{L G I_p} = \frac{T_A L_A}{G I_p} = \frac{T_B L_B}{G I_p}$$

Statically Indeterminate Beams

- Determine the reactions at the supports for the prismatic beam and loading shown.



- Solution



$$M = R_A x - \frac{1}{2} w x^2 \Rightarrow \frac{\partial M}{\partial R_A} = x$$

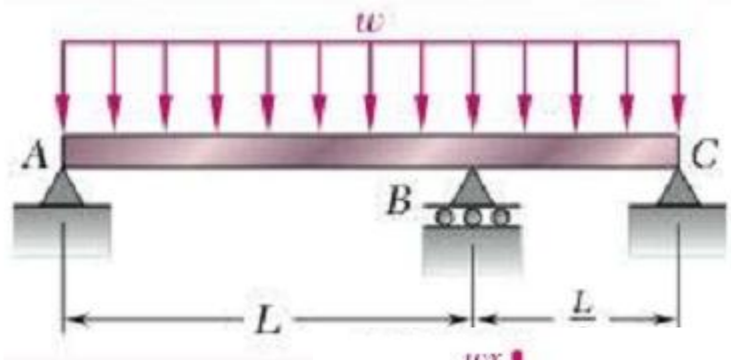
$$0 = y_A = \frac{\partial U}{\partial R_A} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial R_A} dx$$

$$= \frac{1}{EI} \int_0^L \left(R_A x - \frac{1}{2} w x^2 \right) x dx = \frac{1}{EI} \left(\frac{R_A L^3}{3} - \frac{w L^4}{8} \right)$$

$$\Rightarrow R_A = \frac{3}{8} wL \quad \uparrow \qquad \Rightarrow \begin{cases} R_B = \frac{5}{8} wL & \uparrow \\ M_B = \frac{1}{8} wL^2 & \uparrow\downarrow \end{cases}$$

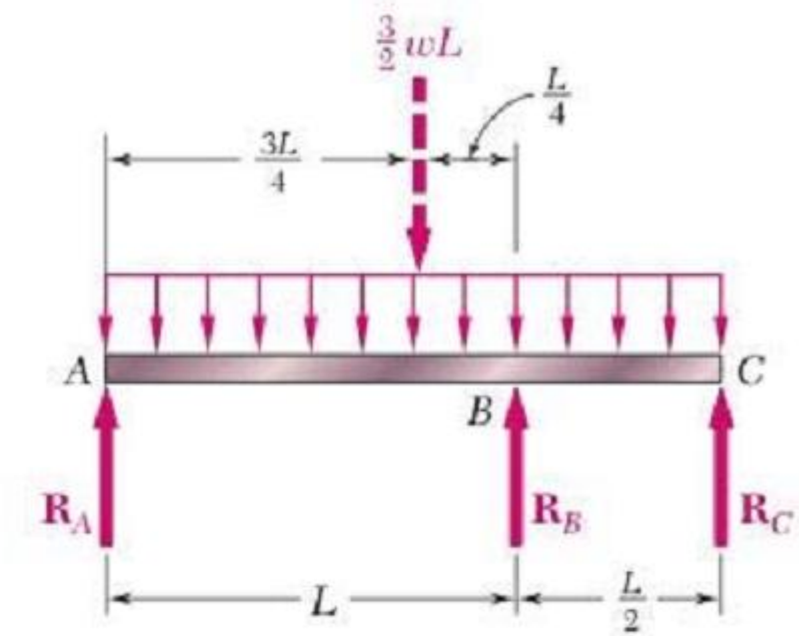
Statically Indeterminate Beams

- For the uniform beam and loading shown, determine the reactions at the supports.



Solution:

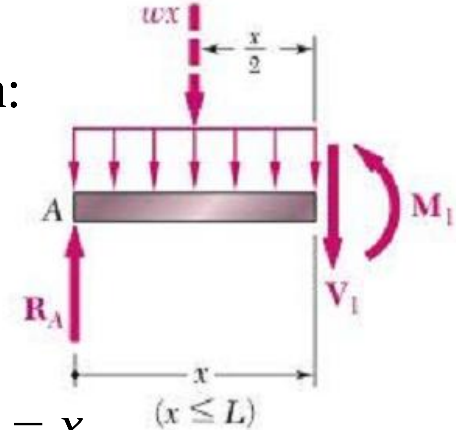
1. Basic determinate system:



2. From static equilibrium:

$$R_B = \frac{9}{4}wL - 3R_A; R_C = -\frac{3}{4}wL + 2R_A$$

3. Portion *AB* of the beam:



$$M_1 = R_A x - \frac{1}{2}wx^2 \Rightarrow \frac{\partial M_1}{\partial R_A} = x$$

$$\frac{1}{EI}\int_0^L M_1 \frac{\partial M_1}{\partial R_A} dx$$

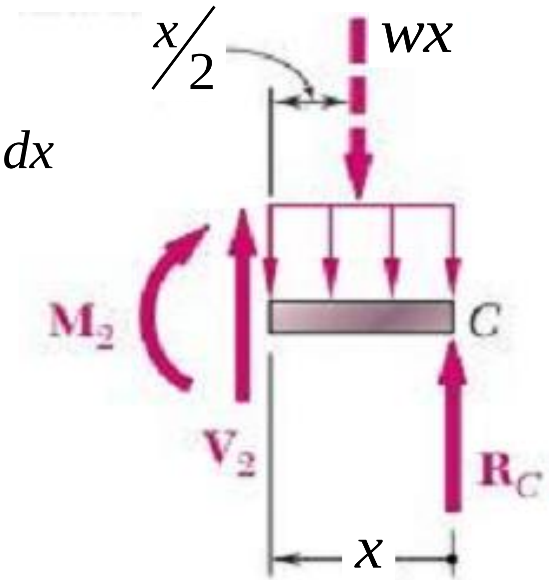
$$= \frac{1}{EI}\int_0^L \left(R_A x - \frac{1}{2}wx^2\right) x dx = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right)$$

4. Portion *CB* of the beam:

$$M_2 = \left(2R_A - \frac{3}{4}wL\right)x - \frac{1}{2}wx^2 \Rightarrow \frac{\partial M_2}{\partial R_A} = 2x$$

$$\frac{1}{EI}\int_0^L M_2 \frac{\partial M_2}{\partial R_A}dx = \frac{1}{EI}\int_0^{L/2}\left(2R_A x - \frac{3}{4}wLx - \frac{1}{2}wx^2\right)(2x)\,dx$$

$$= \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{5wL^4}{64}\right)$$



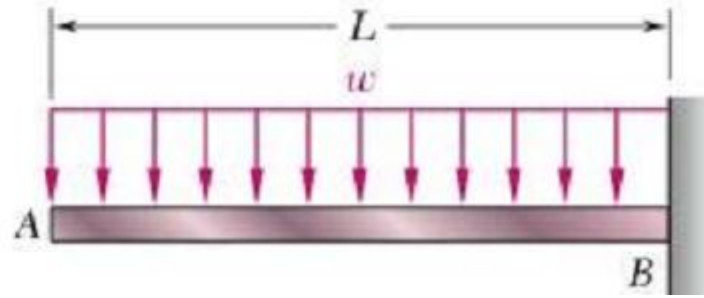
5. Reaction at *A*:

$$0 = y_A = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right) + \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{5wL^4}{64}\right)$$

$$\Rightarrow R_A = \frac{13}{32}wL \quad \uparrow \qquad \Rightarrow \begin{cases} R_B = \dfrac{9}{4}wL - 3R_A = \dfrac{33}{32}wL & \uparrow \\ R_C = -\dfrac{3}{4}wL + 2R_A = \dfrac{1}{16}wL & \uparrow \end{cases}$$

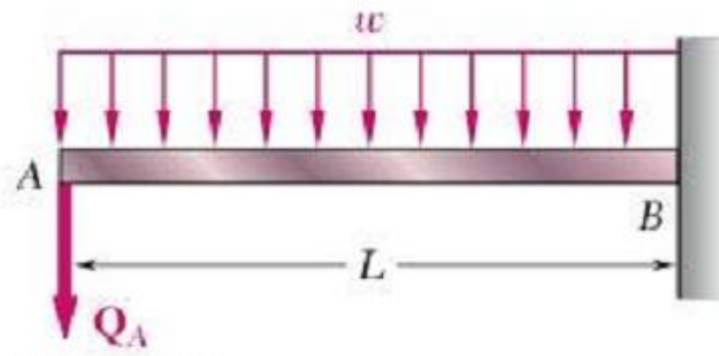
Method of Dummy Load

- The cantilever beam *AB* supports a uniformly distributed load *w*. Determine the deflection and slope at *A*.



Solution:

1. Apply a dummy force Q_A at A:



$$M = -Q_A x - \frac{1}{2}wx^2 \Rightarrow \frac{\partial M}{\partial Q_A} = -x$$

$$y_A = \frac{\partial U}{\partial Q_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial A_A}dx$$

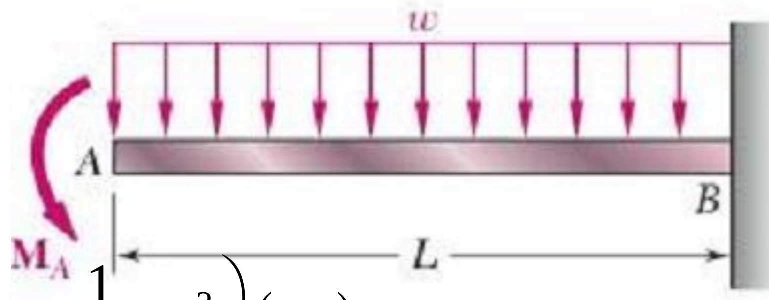
$$= \frac{1}{EI}\int_0^L \left(-Q_A x - \frac{1}{2}wx^2\right)(-x)\,dx$$

2. Set the dummy force Q_A as zero:

$$y_A = \frac{1}{EI}\int_0^L \left(-\cancel{Q_A} x - \frac{1}{2}wx^2\right)(-x)\,dx = \frac{1}{EI}\int_0^L \left(-\frac{1}{2}wx^2\right)(-x)\,dx = +\frac{wL^4}{8EI} \quad \downarrow$$

Note: since the dummy load points downward, + indicates downward deflection at *A*.

3. Apply a dummy moment M_A at A:



$$M = -M_A - \frac{1}{2}wx^2 \Rightarrow \frac{\partial M}{\partial M_A} = -1$$

$$\theta_A = \frac{\partial U}{\partial M_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_A}dx = \frac{1}{EI}\int_0^L\left(-M_A - \frac{1}{2}wx^2\right)(-1)\,dx$$

4. Set the dummy moment M_A as zero:

$$\theta_A = \frac{1}{EI}\int_0^L\left(-\cancel{M_A} - \frac{1}{2}wx^2\right)(-1)\,dx = \frac{1}{EI}\int_0^L\left(-\frac{1}{2}wx^2\right)(-1)\,dx = +\frac{wL^3}{6EI} \quad \downarrow\uparrow$$

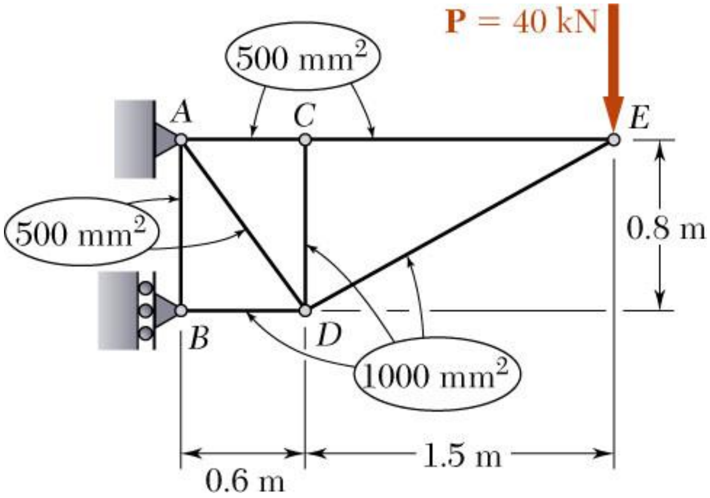
Note: since the dummy moment acts counter clockwise, + indicates counter clockwise rotation of cross-section A.

$$y_A = \frac{1}{EI}\int_0^L\left(-\frac{1}{2}wx^2\right)(-x)\,dx = \frac{1}{EI}\int_0^L M\bar{M}dx; \quad \theta_A = \frac{1}{EI}\int_0^L\left(-\frac{1}{2}wx^2\right)(-1)\,dx = \frac{1}{EI}\int_0^L M\bar{M}dx$$

M : bending moment in beam developed by real loads.

$\bar{M}$: fictitious moment in beam developed by a unit dummy load (force/moment) applied at a point of interest.

Sample Problem

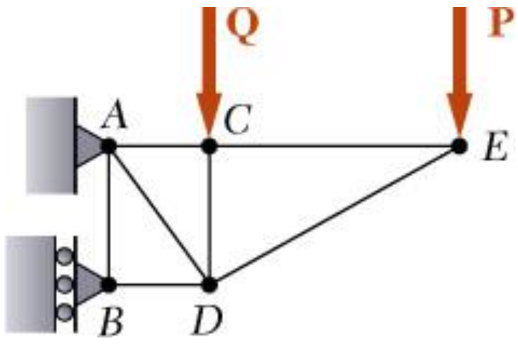


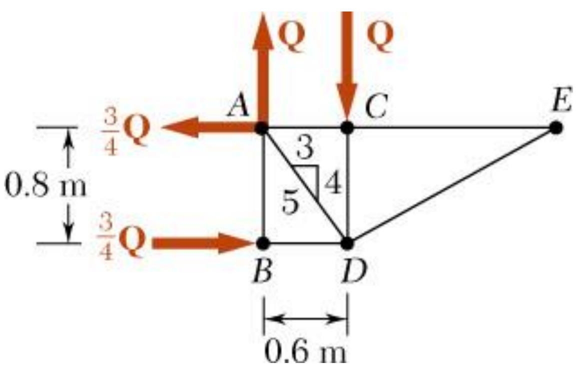
Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using E = 73 GPa, determine the vertical deflection of the joint C caused by the load P.

Solution:

- For application of Castigliano's theorem, introduce a dummy vertical load Q at C. Find the reactions at A and B due to the dummy load from a free-body diagram of the entire truss.
- Apply the method of joints to determine the axial force in each member due to Q.
- Combine with the results of previous example to evaluate the derivative with respect to Q of the strain energy of the truss due to the loads P and Q.
- Setting $Q = 0$, evaluate the derivative which is equivalent to the desired displacement at C.

Solution:

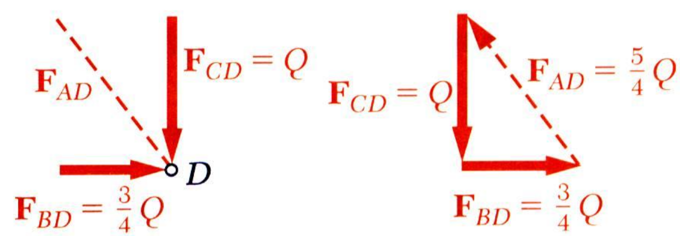




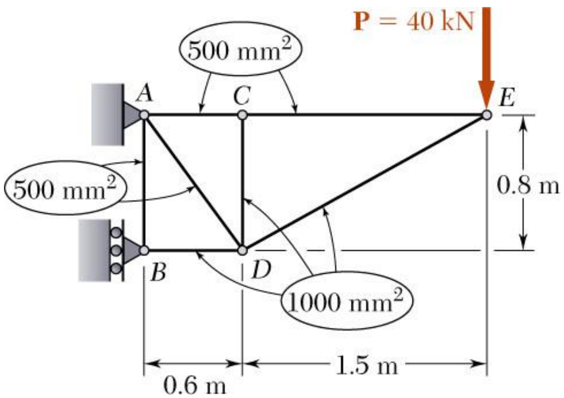
- Find the reactions at A and B due to a dummy load Q at C from a free-body diagram of the entire truss.

$$A_x = -\tfrac{3}{4}Q \qquad A_y = Q \qquad B = \tfrac{3}{4}Q$$

- Apply the method of joints to determine the axial force in each member due to Q.



$$F_{CE} = F_{DE} = 0$$
$$F_{AC} = 0;\ F_{CD} = -Q$$
$$F_{AB} = 0;\ F_{BD} = -\tfrac{3}{4}Q;\ F_{AD} = \tfrac{5}{4}Q$$



Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4+5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P+3125Q$
BD	$-21P/8-3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P+338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

- Combine with the results of previous example to evaluate the derivative with respect to *Q* of the strain energy of the truss due to the loads *P* and *Q*.

$$y_C=\sum\frac{F_iL_i}{EA_i}\frac{\partial F_i}{\partial Q}=\sum\frac{(P_i+Q_i)L_i}{EA_i}\frac{\partial(P_i+Q_i)}{\partial Q}=\frac{1}{E}(4306P+4263Q)$$

- Setting *Q* = 0, evaluate the derivative which is equivalent to the desired displacement at *C*.

$$y_C=\sum\frac{P_iL_i}{EA_i}\left[\frac{\partial Q_i}{\partial Q}\right]_{Q=0}=\frac{4306\left(40\times10^3\,N\right)}{73\times10^9\,\mathrm{Pa}}$$

$$y_C=2.36\,\mathrm{mm}\downarrow$$

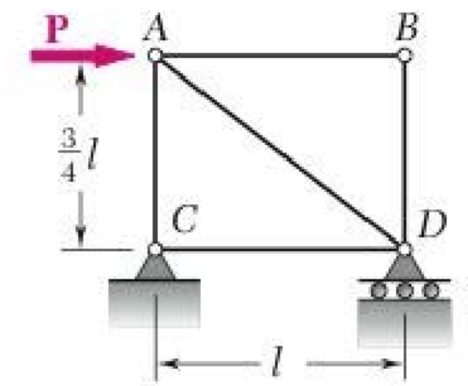
$$\left[\frac{\partial Q_i}{\partial Q}\right]_{Q=0}=\bar{F}_i$$: axial forces developed in individual members under a unit load applied at joint C.

$$y_C=\sum\frac{F_iL_i}{A_iE}\bar{F}_i$$

Method of Unit Dummy Load

- For an elastic structure, the deflection of a particular point can be found by applying a unit dummy load at the point of interest

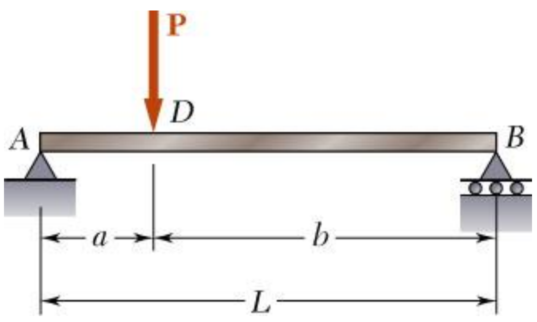
- Tension/compression



$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2EA_i}$$

$$y = \sum \frac{F_i L_i}{EA_i} \bar{F}_i$$

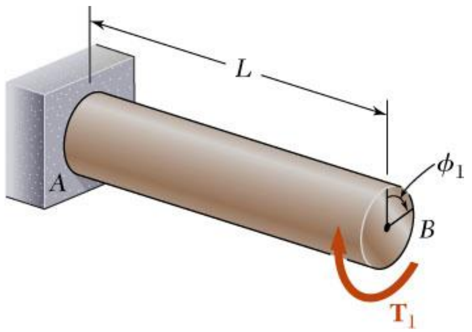
- Bending



$$U = \int_0^L \frac{M^2}{2EI} dx$$

$$y = \int_0^L \frac{M\bar{M}}{EI} dx$$

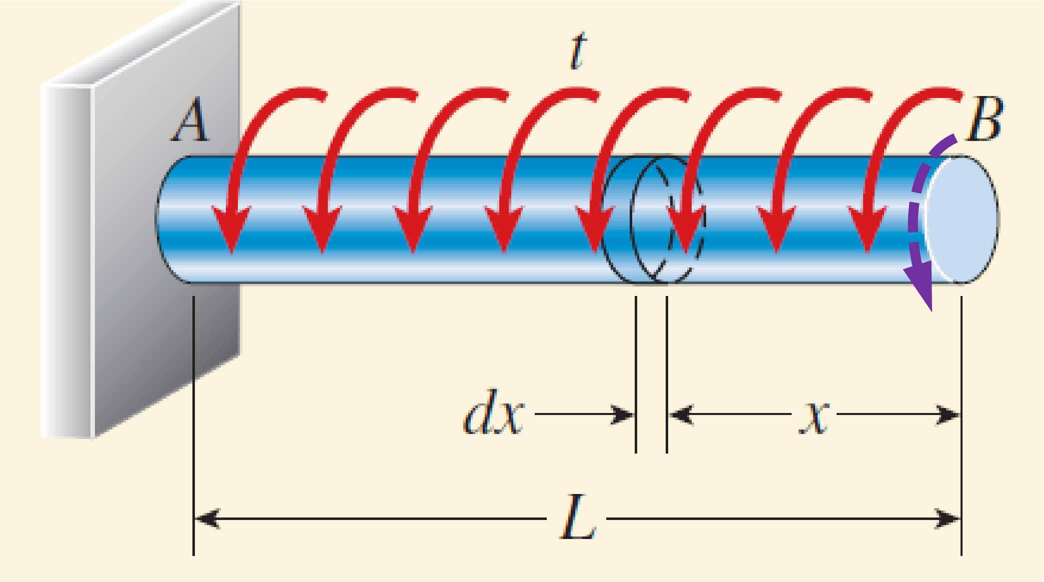
- Torsion



$$U = \int_0^L \frac{T^2}{2GI_p} dx$$

$$\phi = \int_0^L \frac{T\bar{T}}{GI_p} dx$$

Sample Problem



- Determine the angle of twist at cross-section *B* of the shaft.

- Solution

$$\phi_B = \int_0^L \frac{T\overline{T}}{GI_p}dx = \int_0^L \frac{(tx)(1)}{GI_p}dx = \frac{tL^2}{2GI_p}$$

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