Hypo-elastic Materials

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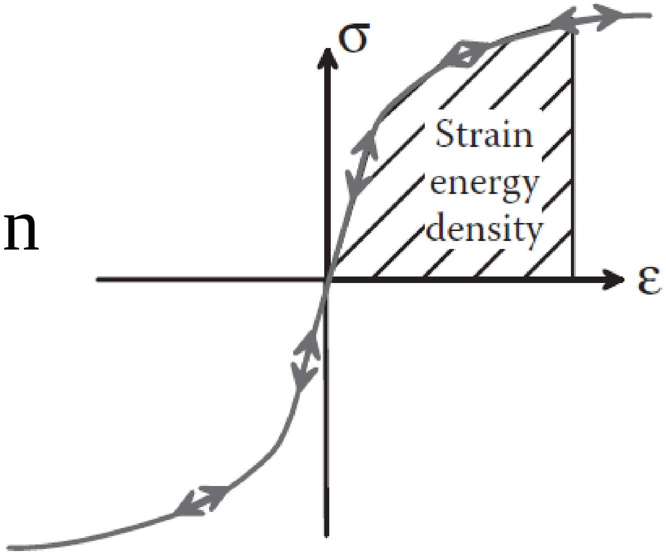
Outline

- Introduction（引言）
- Nonlinear elastic shear model（非线性弹性剪切模型）
- Calibration of nonlinear elastic shear model（模型校准）

Introduction

- Hypoelasticity is used to model materials that exhibit nonlinear, but reversible, stress-strain behavior even at small strains.
- The specimen deforms reversibly: if you remove the loads, the solid returns to its original shape.



- The strain in the specimen depends only on the stress applied to it; it does not depend on the rate of loading or the history of loading.
- We will assume here that the material is isotropic.

Introduction

- Strains and rotations are assumed to be small. We use infinitesimal strain and Cauchy stress.
- Existence of a strain energy density guarantees that deformations of the material are perfectly reversible.
- If the material is isotropic, the strain energy can only be a function of invariants of the strain tensor, i.e. three principal strains.
- It is usually more convenient to use the three fundamental scalar invariants:

$$\varepsilon_{ij} n_j = \varepsilon_n n_i$$

$$\det\left[\varepsilon_{ij} - \varepsilon_n \delta_{ij}\right] = 0$$

$$\boxed{-\varepsilon_n^{\ 3} + I_1 \varepsilon_n^{\ 2} - I_2 \varepsilon_n + I_3 = 0}$$

$$I_1 = \varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$I_2 = \frac{1}{2}\left(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}\right) = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$I_3 = \det\left[\varepsilon_{ij}\right] = \frac{1}{6} \in_{ijk} \in_{rst} \varepsilon_{ir}\varepsilon_{js}\varepsilon_{kt} = \varepsilon_1\varepsilon_2\varepsilon_3$$

Nonlinear Elastic Shear Model

- In most practical applications, nonlinear behavior is only observed when the material is subjected to shear deformation (characterized by I_2), whereas stress varies linearly with volume changes (characterized by I_1).
- Note the slightly different definition of I_2, which renders us great simplification in deriving the inverse relation:

$$I_1 = \varepsilon_{kk} \Rightarrow \frac{\partial I_1}{\partial \varepsilon_{ij}} = \delta_{ij}, I_2 = \frac{1}{2}\left(\varepsilon_{ij}\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\varepsilon_{ll}\right) \Rightarrow \frac{\partial I_2}{\partial \varepsilon_{ij}} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$U\left[I_1, I_2\right] = \frac{1}{2}KI_1^2 + \frac{2n\sigma_0\varepsilon_0}{n+1}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(n+1)/2n} \Rightarrow \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = KI_1\frac{\partial I_1}{\partial \varepsilon_{ij}} + \frac{\sigma_0}{\varepsilon_0}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(1-n)/2n}\frac{\partial I_2}{\partial \varepsilon_{ij}}$$

$$\Rightarrow \boxed{\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + \frac{\sigma_0}{\varepsilon_0}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(1-n)/2n}\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right)}$$

Nonlinear Elastic Shear Model

- Strain in terms of stress

$$\sigma_{ij} = K\varepsilon_{kk}\delta_{ij} + \frac{\sigma_0}{\varepsilon_0}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(1-n)/2n}\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right) \Rightarrow \sigma_{kk} = 3K\varepsilon_{kk} \Rightarrow \varepsilon_{kk} = \frac{1}{3K}\sigma_{kk}$$

$$\Rightarrow \varepsilon_{ij} = \frac{1}{3}\varepsilon_{kk}\delta_{ij} + \frac{\varepsilon_0}{\sigma_0}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(n-1)/2n}\left(\sigma_{ij} - K\varepsilon_{kk}\delta_{ij}\right)$$

$$\Rightarrow \boxed{\varepsilon_{ij} = \frac{1}{9K}\sigma_{kk}\delta_{ij} + \frac{\varepsilon_0}{\sigma_0}\left(\frac{I_2'}{\sigma_0^2}\right)^{(n-1)/2}\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)}$$

$$\sigma_{ij}\sigma_{ij} = 3K^2\varepsilon_{kk}\varepsilon_{ll} + 2\sigma_0^2\left(\frac{I_2}{\varepsilon_0^2}\right)^{1/n} = \frac{1}{3}\sigma_{kk}\sigma_{ll} + 2\sigma_0^2\left(\frac{I_2}{\varepsilon_0^2}\right)^{1/n}$$

$$\Rightarrow \left(\frac{I_2}{\varepsilon_0^2}\right)^{1/n} = \frac{1}{2\sigma_0^2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{ll}\right) = \frac{I_2'}{\sigma_0^2}, I_2' = \frac{1}{2}\left(\sigma_{ij}\sigma_{ij} - \frac{1}{3}\sigma_{kk}\sigma_{ll}\right)$$

Calibration of Nonlinear Elastic Shear Model

$$\varepsilon_{ij}=\frac{1}{9K}\sigma_{kk}\delta_{ij}+\frac{\varepsilon_0}{\sigma_0}\left(\frac{I_2'}{\sigma_0^2}\right)^{(n-1)/2}\left(\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}\right),\qquad I_2'=\frac{1}{2}\left(\sigma_{ij}\sigma_{ij}-\frac{1}{3}\sigma_{kk}\sigma_{ll}\right)$$

- Hydrostatic stress state

$$\sigma_{11}=\sigma_{22}=\sigma_{33}=p\Rightarrow\varepsilon_{11}=\varepsilon_{22}=\varepsilon_{33}=\frac{1}{9K}(3p)+\frac{\varepsilon_0}{\sigma_0}\left(\frac{I_2'}{\sigma_0^2}\right)^{(n-1)/2}\left(p-\frac{1}{3}(3p)\right)=\frac{p}{3K}$$

- Pure shear

$$\sigma_{12}=\sigma_{21}=\tau\quad\Rightarrow\varepsilon_{12}=\varepsilon_{21}=\frac{1}{9K}\sigma_{kk}\delta_{12}+\frac{\varepsilon_0}{\sigma_0}\left(\frac{\tau^2}{\sigma_0^2}\right)^{(n-1)/2}\left(\tau-\frac{1}{3}\sigma_{kk}\delta_{12}\right)=\varepsilon_0\left(\frac{\tau}{\sigma_0}\right)^n$$

- Uniaxial tension

$$\sigma_{11}=\sigma, I_2'=\frac{1}{2}\left(\sigma^2-\frac{1}{3}\sigma^2\right)=\frac{\sigma^2}{3}\Rightarrow\begin{cases}\varepsilon_{11}=\frac{1}{9K}\sigma\delta_{11}+\frac{\varepsilon_0}{\sigma_0}\left(\frac{\sigma^2}{3\sigma_0^2}\right)^{(n-1)/2}\left(\sigma-\frac{1}{3}\sigma\delta_{11}\right)=\frac{\sigma}{9K}+\frac{2\varepsilon_0}{\sqrt{3}}\left(\frac{\sigma}{\sqrt{3}\sigma_0}\right)^n\\ \varepsilon_{22}=\varepsilon_{33}=\frac{1}{9K}\sigma\delta_{22}+\frac{\varepsilon_0}{\sigma_0}\left(\frac{\sigma^2}{3\sigma_0^2}\right)^{(n-1)/2}\left(-\frac{1}{3}\sigma\delta_{22}\right)=\frac{\sigma}{9K}-\frac{\varepsilon_0}{\sqrt{3}}\left(\frac{\sigma}{\sqrt{3}\sigma_0}\right)^n\end{cases}$$