



Statically Indeterminate Structures

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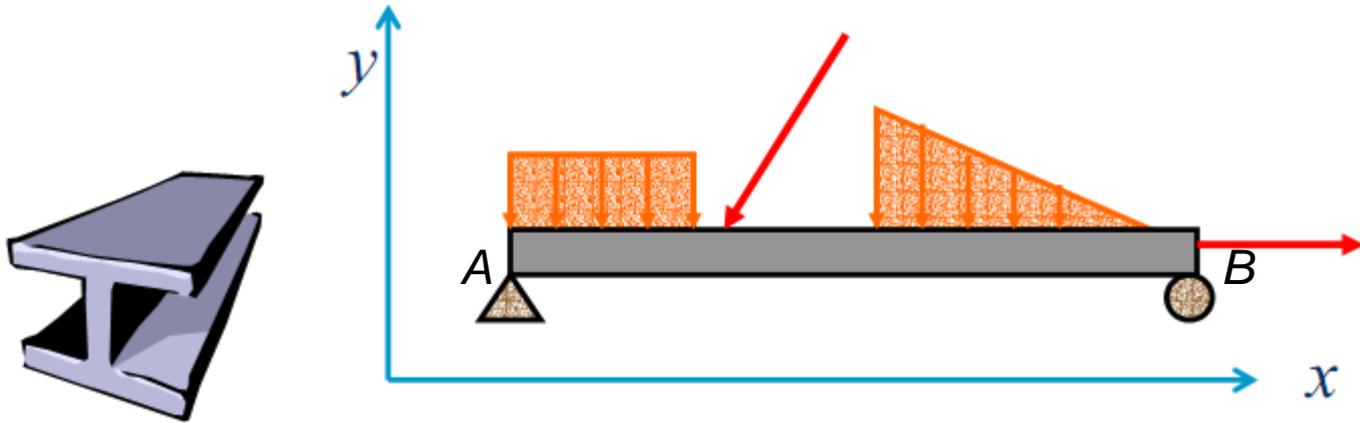
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Statically Determinate Problems

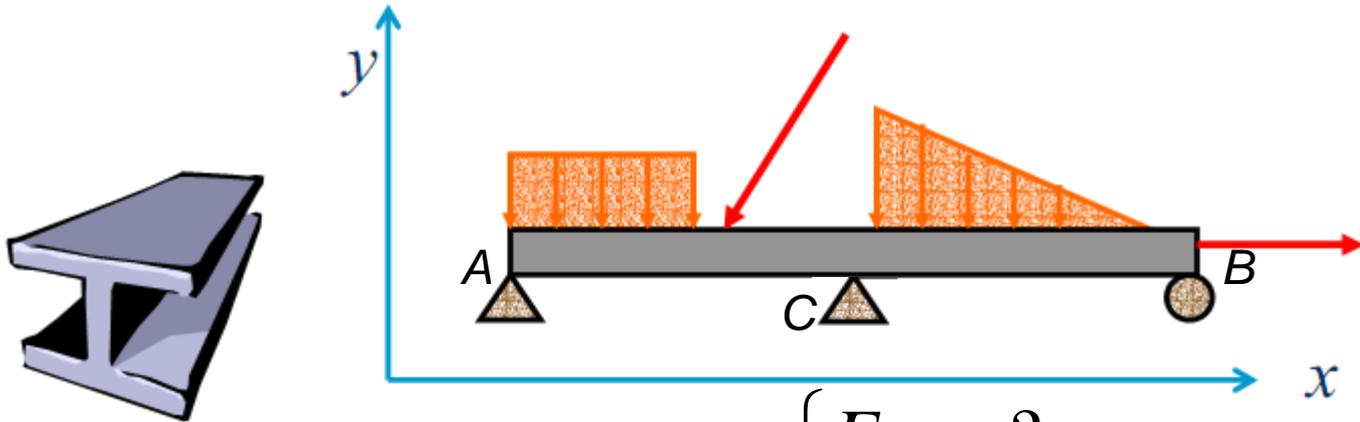
- Problems can be completely solved via static equilibrium
- Number of unknowns (forces/moments) = number of independent equations from static equilibration



$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_A = 0 \end{cases} \Rightarrow \begin{cases} F_{Ax} = ? \\ F_{Ay} = ? \\ F_{By} = ? \end{cases}$$

Statically Indeterminate Problems

- Problems cannot be solved from static equilibrium alone
- Number of unknowns (forces/moments) > number of independent equations from static equilibration



$$\left\{ \begin{array}{l} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_A = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} F_{Ax} = ? \\ F_{Ay} = ? \\ F_{By} = ? \\ F_{Cx} = ? \\ F_{Cy} = ? \end{array} \right.$$

Degrees of Indeterminacy of Structures

- For a co-planer structure, there are at most three equilibrium equations for each portion of the structure. If there is a total of n portions and m unknown reaction forces from supports:

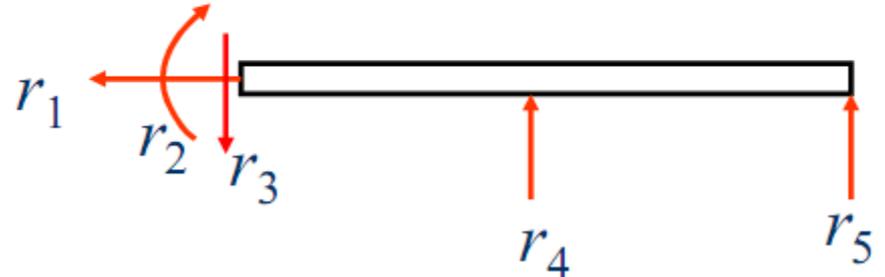
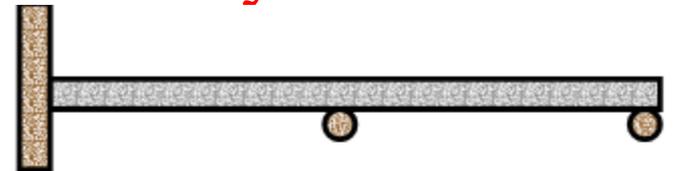
$m = 3n \Rightarrow$ Statically determinate

$m > 3n \Rightarrow$ Statically indeterminate

$m - 3n =$ Degrees of indeterminacy



Statically determinate



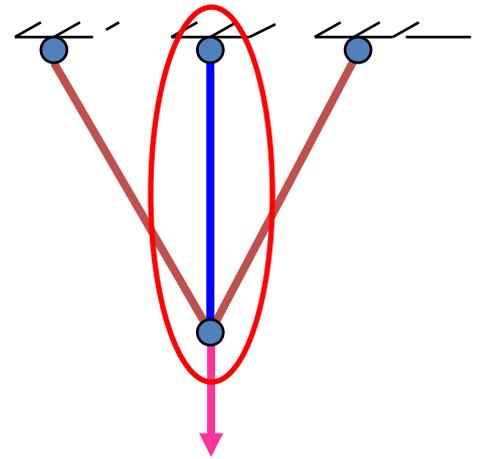
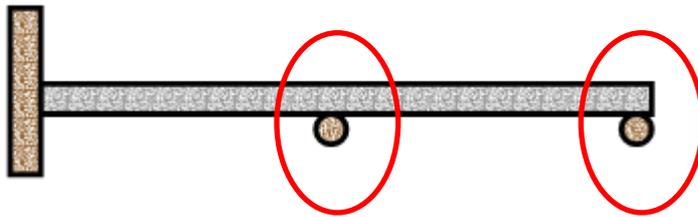
Degrees of Indeterminacy = $\frac{2}{6}$

Advantages & Disadvantages of Indeterminate Structures

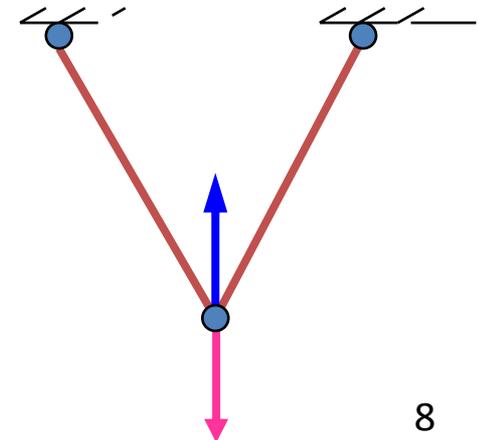
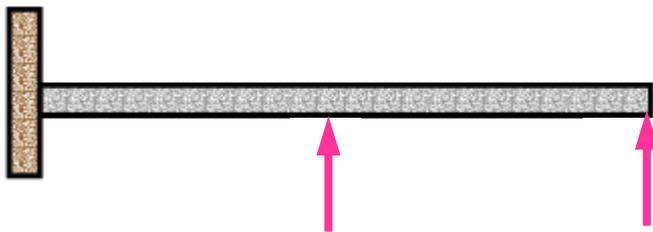
- Advantages:
 - Redistribution of reaction forces / internal forces
 - Smaller deformation
 - Greater stiffness as a whole structure
- Disadvantages:
 - Thermal and residual stresses due to temperature change and fabrication errors

Redundancy & Basic Determinate System

- **Redundancy:** unnecessary restraints without which the static equilibrium of a structure still holds.



- **Basic determinate system:** the same structure as of a statically indeterminate system after replacing redundant restraints with extra constraining loads

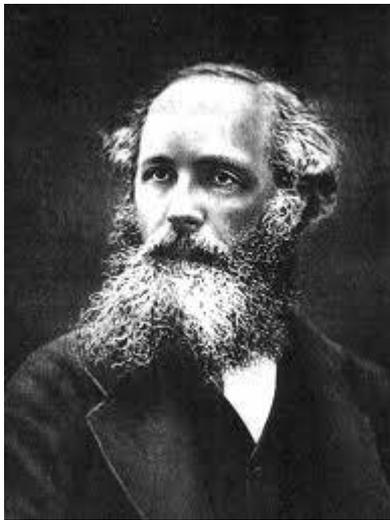


How to Analyze Statically Indeterminate Structures?

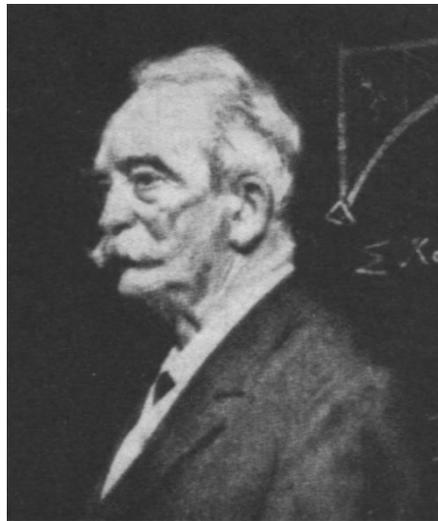
- **Basic determinate structure:** obtained via replacing redundant restraints with extra constraining loads.
- **Equilibrium:** is satisfied when the reaction forces at supports hold the structure at rest, as the structure is subjected to external loads
- **Deformation compatibility:** satisfied when the various segments of the structure fit together without intentional breaks or overlaps
- **Deformation-load relationship:** depends on the manner the material of the structure responds to the applied loads, which can be linear/nonlinear/viscous and elastic/inelastic; for our study the behavior is assumed to be linearly elastic

Force Method (Method of Consistent Deformations)

- The method of consistent deformations or force method was originally developed by James Clerk Maxwell in 1874 and later refined by Otto Mohr and Heinrich Müller-Breslau.



James Clerk Maxwell
(1831 – 1879)



Christian Otto Mohr
(1835 – 1918)



Heinrich Müller-Breslau
(1851 - 1925)

Force Method Solution Procedure

- Making the indeterminate structure determinate
- Obtaining its equations of equilibrium
- Analyzing the deformation compatibility that must be satisfied at selected redundant restraints
- Substituting the deformation-load relationship into deformation compatibility, resulting in complementary equations
- Joining of equilibrium equations and complementary equations
- *Types of Problems can be addressed include (a) bar tension/compression; (b) shaft torsion; and (c) beam bending*

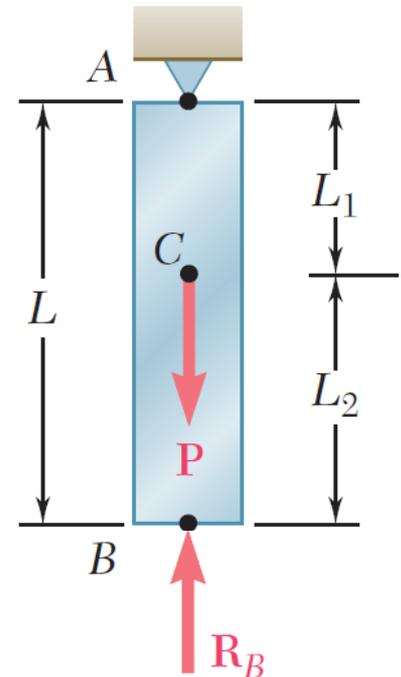
Statically Indeterminate Bars

- For an axially loaded bar with both ends fixed, determine the reaction forces.
- Solution:

$$0 = \Delta L = -\frac{R_B L_2}{EA} + \frac{(P - R_B) L_1}{EA}$$

$$\Rightarrow R_B = \frac{L_1}{L} P$$

$$\Rightarrow R_A = \frac{L_2}{L} P$$



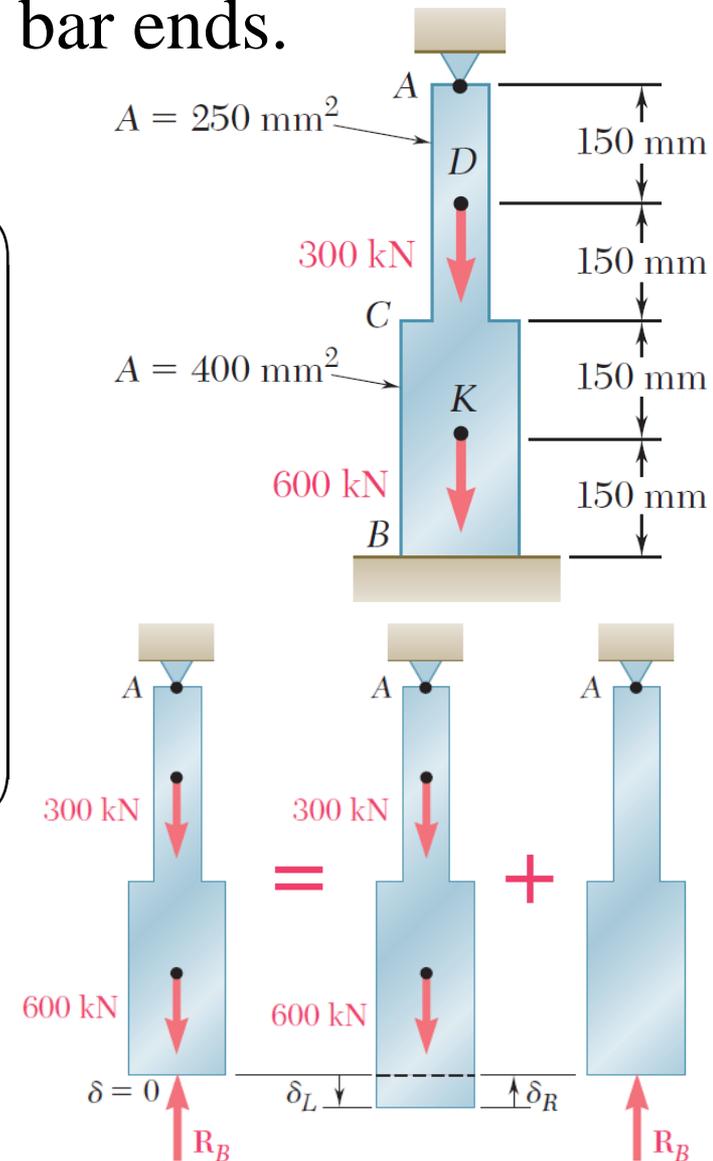
Sample Problem

- Determine the reaction forces at the bar ends.
- Solution:

$$0 = \Delta L = \sum_{i=1}^4 \frac{F_i L_i}{EA_i} = \frac{150 \times 10^{-3}}{E} \left(\begin{array}{l} -\frac{R_B}{400 \times 10^{-6}} \\ + \frac{(600 - R_B) \times 10^3}{400 \times 10^{-6}} \\ + \frac{(600 - R_B) \times 10^3}{250 \times 10^{-6}} \\ + \frac{(900 - R_B) \times 10^3}{250 \times 10^{-6}} \end{array} \right)$$

$$\Rightarrow R_B = 577 \text{ kN}$$

$$\Rightarrow R_A = 900 - 577 = 323 \text{ kN}$$



Sample Problem

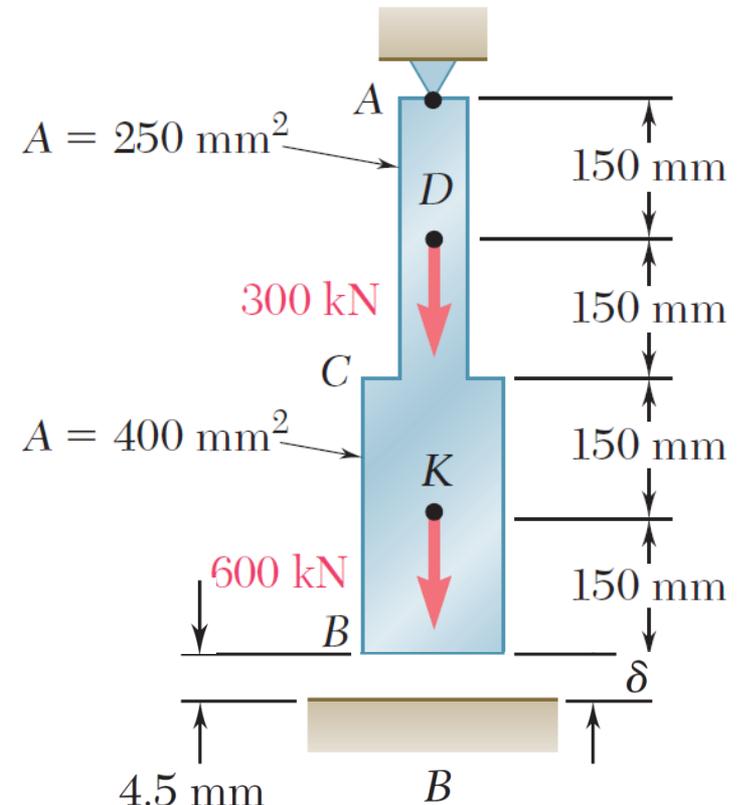
- Determine the reaction forces, if any, at the bar ends. $E = 200 \text{ GPa}$.
- Solution:

$$4.5 \times 10^{-3} = \Delta L = \sum_{i=1}^4 \frac{F_i L_i}{EA_i}$$

$$= \frac{150 \times 10^{-3}}{200 \times 10^9} \left(\begin{array}{l} -\frac{R_B}{400 \times 10^{-6}} \\ +\frac{(600 - R_B) \times 10^3}{400 \times 10^{-6}} \\ +\frac{(600 - R_B) \times 10^3}{250 \times 10^{-6}} \\ +\frac{(900 - R_B) \times 10^3}{250 \times 10^{-6}} \end{array} \right)$$

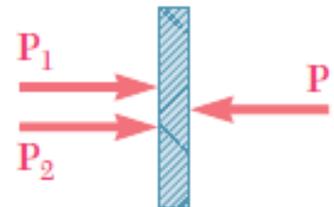
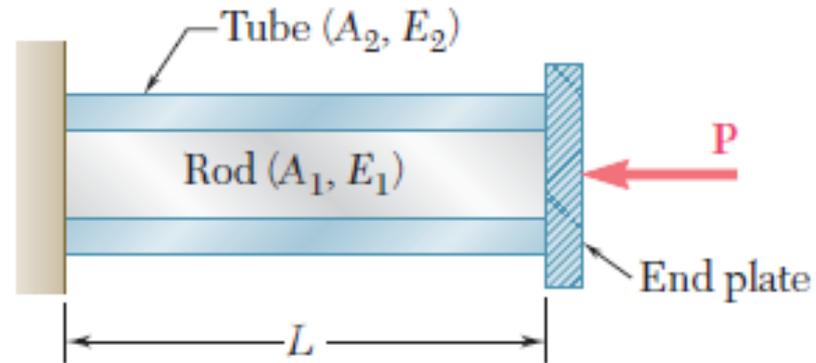
$$\Rightarrow R_B = 115.4 \text{ kN}$$

$$\Rightarrow R_A = 900 - 115.4 = 784.6 \text{ kN}$$



Sample Problem

- What is the deformation of the rod and tube when a force P is exerted on a rigid end plate as shown?
- Solution:



$$\begin{cases} P_1 + P_2 = P \\ \Delta L_1 = \frac{P_1 L}{E_1 A_1} = \frac{P_2 L}{E_2 A_2} = \Delta L_2 \end{cases}$$

$$\Rightarrow P_1 = \frac{E_1 A_1 P}{E_1 A_1 + E_2 A_2}, \quad P_2 = \frac{E_2 A_2 P}{E_1 A_1 + E_2 A_2}$$

$$\Rightarrow \Delta L = \frac{PL}{E_1 A_1 + E_2 A_2}$$

Sample Problem

- Given: tension/compression rigidities $E_1A_1 = E_2A_2 = EA$, E_3A_3 , external load F . Find: internal forces in bars.

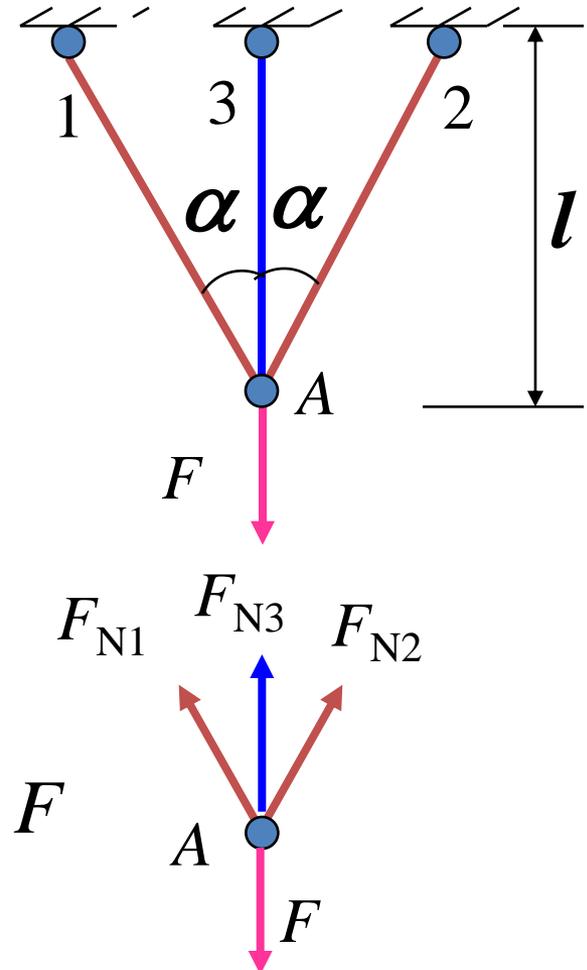
- Solution:

1. Make the truss determinate by replacing extension/contraction restraint along Bar 3 with F_{N3}

2. Static equilibrium at joint A

$$\sum F_x = 0 \Rightarrow F_{N1} = F_{N2}$$

$$\sum F_y = 0 \Rightarrow (F_{N1} + F_{N2}) \cos \alpha + F_{N3} = F$$

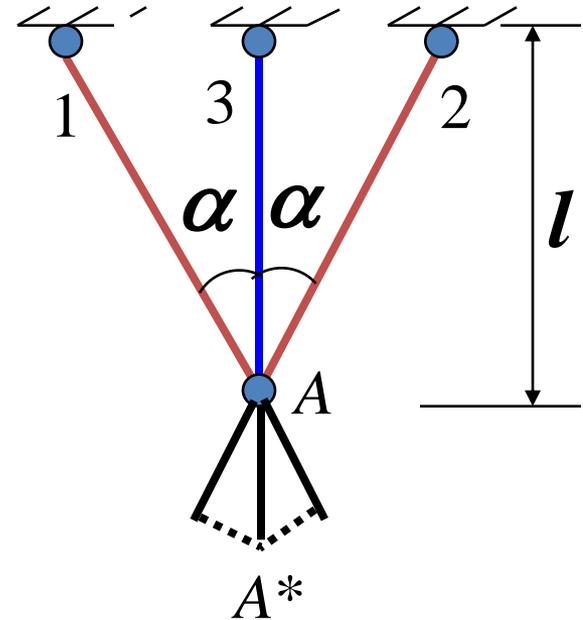


3. Displacement compatibility at joint A

Due to symmetry

$$\Delta l_1 = \Delta l_2$$

$$\Delta l_1 = \Delta l_3 \cos \alpha$$



4. Displacement-force relationship

$$\Delta l_1 = \frac{F_{N1} l}{EA \cos \alpha} \quad \Delta l_3 = \frac{F_{N3} l}{E_3 A_3}$$

5. Complementary equation

$$\Delta l_1 = \frac{F_{N1} l}{E_1 A_1 \cos \alpha} = \Delta l_3 \cos \alpha = \frac{F_{N3} l}{E_3 A_3} \cos \alpha \Rightarrow F_{N1} = F_{N3} \frac{EA}{E_3 A_3} \cos^2 \alpha$$

6. Joint of force equilibrium and complementary equations

$$\left\{ \begin{array}{l} F_{N1} = F_{N2} \\ (F_{N1} + F_{N2}) \cos \alpha + F_{N3} = F \\ F_{N1} = F_{N3} \frac{EA}{E_3 A_3} \cos^2 \alpha \end{array} \right. \Rightarrow \left\{ \begin{array}{l} F_{N1} = F_{N2} = \frac{F}{2 \cos \alpha + \frac{E_3 A_3}{EA \cos^2 \alpha}} \\ F_{N3} = \frac{F}{1 + 2 \frac{EA}{E_3 A_3} \cos^3 \alpha} \end{array} \right.$$

Sample Problem

- Given: tension/compression rigidities E_1A_1 , E_2A_2 , E_3A_3 , external load F . Find: internal forces in bars.

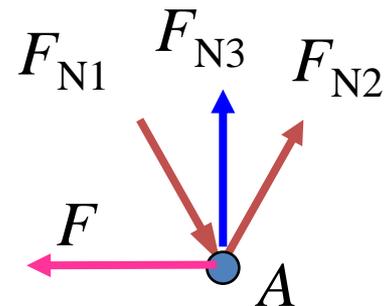
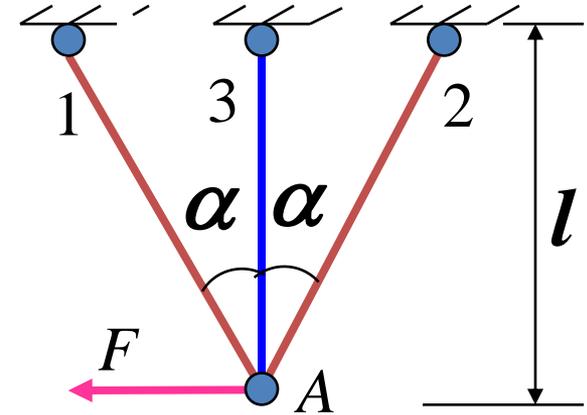
- Solution:

1. Make the truss determinate by replacing extension/contraction restraint along Bar 3 with F_{N3}

2. Static equilibrium at joint A

$$F_{N1} \sin \alpha + F_{N2} \sin \alpha = F \quad (1)$$

$$F_{N1} \cos \alpha - F_{N2} \cos \alpha - F_{N3} = 0 \quad (2)$$



3. Displacement compatibility at joint A

$$\frac{\Delta l_2}{\sin \alpha} = \frac{\Delta l_1}{\sin \alpha} + 2 \frac{\Delta l_3}{\tan \alpha}$$

$$\Rightarrow \Delta l_2 = \Delta l_1 + 2\Delta l_3 \cos \alpha$$

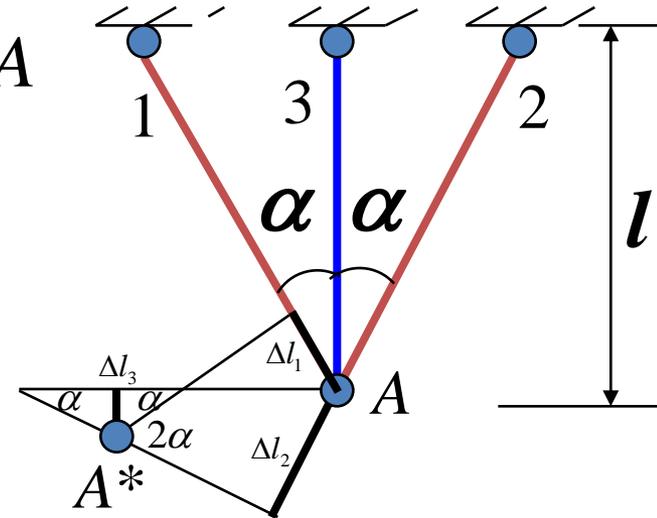
4. Complementary equation

$$\Delta l_1 = \frac{F_{N1} l}{E_1 A_1 \cos \alpha}, \quad \Delta l_2 = \frac{F_{N2} l}{E_2 A_2 \cos \alpha}, \quad \Delta l_3 = \frac{F_{N3} l}{E_3 A_3}$$

$$\Rightarrow \frac{F_{N2} l}{E_2 A_2 \cos \alpha} = \frac{F_{N1} l}{E_1 A_1 \cos \alpha} + 2 \frac{F_{N3} l}{E_3 A_3} \cos \alpha$$

$$\Rightarrow \frac{F_{N1}}{E_1 A_1} - \frac{F_{N2}}{E_2 A_2} + 2 \frac{F_{N3}}{E_3 A_3} \cos^2 \alpha = 0 \quad (3)$$

5. Joint of force equilibrium and complementary equations



Assembly Stress

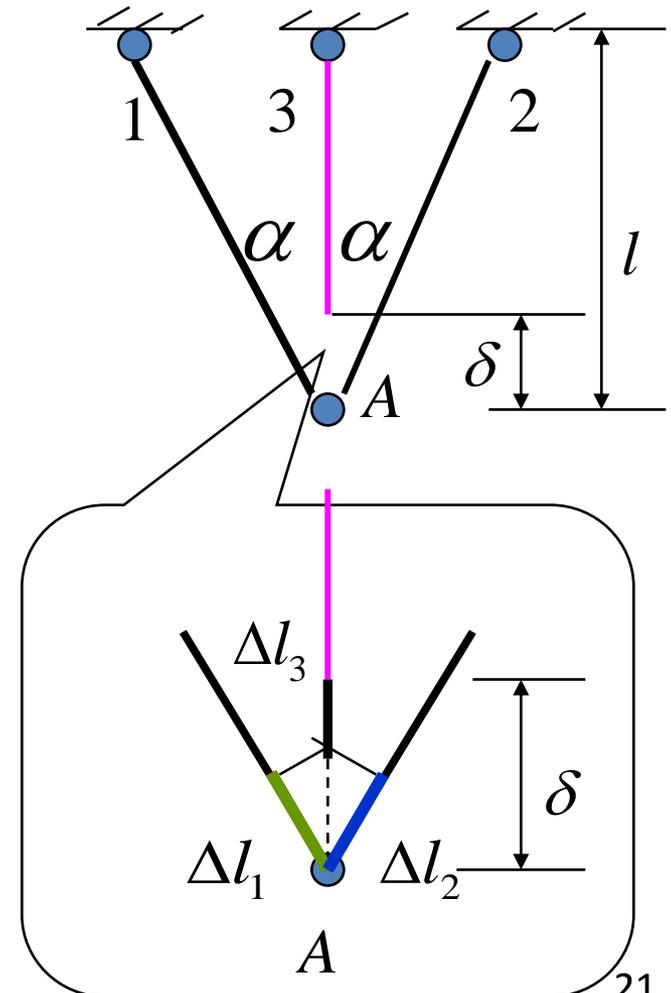
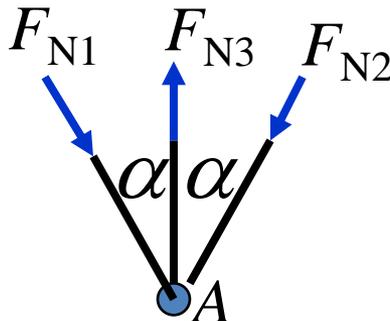
- Residual stresses exist in statically indeterminate structures, originated from assembling / fabrication errors of structural elements
- Bar 3 is δ shorter than required
- Force equilibrium:

$$F_{N1} = F_{N2}$$

$$F_{N3} = (F_{N1} + F_{N2}) \cos \alpha$$

- Displacement compatibility:

$$\Delta l_3 + \frac{|\Delta l_1|}{\cos \alpha} = \delta$$



Sample Problem

- Given: AB rigid; $E_1A_1 = E_2A_2 = EA$; Bar 1 is δ shorter than required. Find: internal forces in Bar 1 and 2 after assembly.
- Solution:

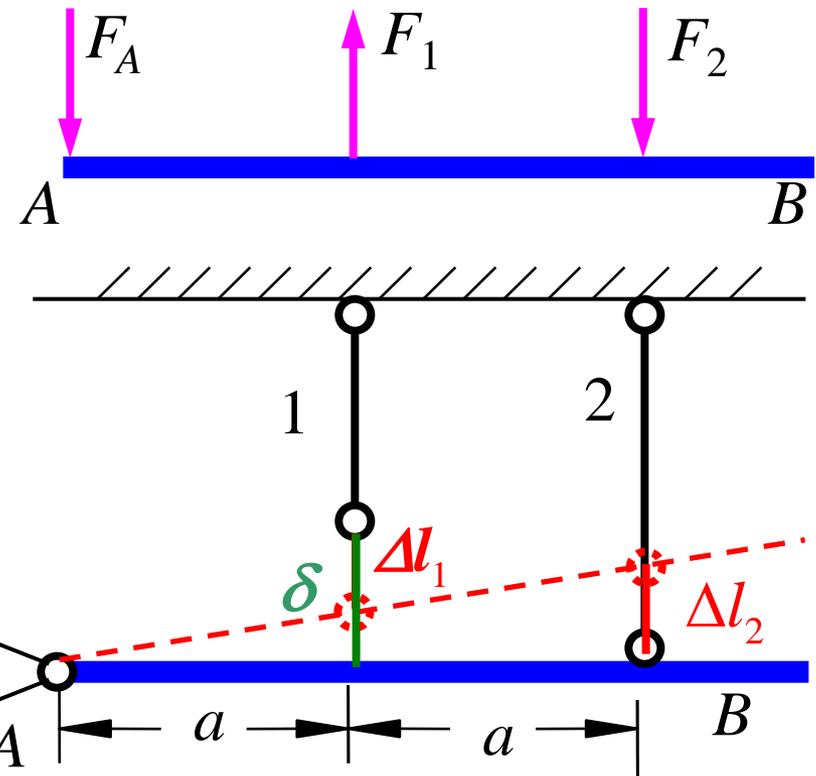
1. Basic determinate structure and force equilibrium of AB :

$$\sum M_A = 0 \Rightarrow aF_1 = 2aF_2$$

2. Displacement compatibility:

Bar 1 lengthened & Bar 2 shortened:

$$\frac{\delta - \Delta l_1}{\Delta l_2} = \frac{a}{2a} \Rightarrow 2(\delta - \Delta l_1) = \Delta l_2$$



3. Displacement-force relationship:

$$\Delta l_1 = \frac{F_1 l}{EA}; \quad \Delta l_2 = \frac{F_2 l}{EA}$$

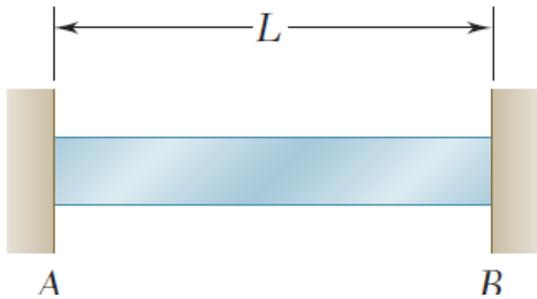
4. Complementary equation:

$$2(\delta - \Delta l_1) = \Delta l_2 \Rightarrow 2\left(\delta - \frac{F_1 l}{EA}\right) = \frac{F_2 l}{EA}$$

5. Joint of force equilibrium and complementary equations

$$\begin{cases} aF_1 = 2aF_2 \\ 2\left(\delta - \frac{F_1 l}{EA}\right) = \frac{F_2 l}{EA} \end{cases} \Rightarrow \begin{cases} F_1 = \frac{4\delta EA}{5l} \\ F_2 = \frac{2\delta EA}{5l} \end{cases}$$

Thermo Stress



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.
- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha (\Delta T) L$$

$$\delta_P = \frac{PL}{AE}$$

α = thermal expansion coefficient

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha (\Delta T)$$

$$\alpha (\Delta T) L + \frac{PL}{AE} = 0$$

$$\sigma = \frac{P}{A} = -E\alpha (\Delta T)$$

- Example (low-carbon steel)

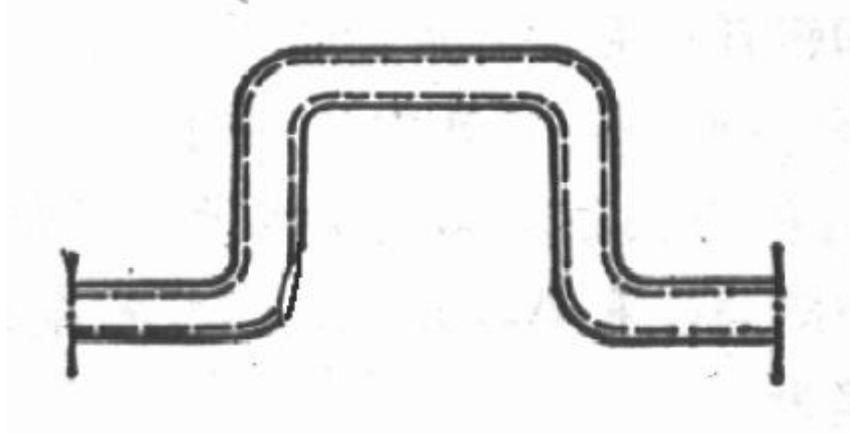
$$\sigma = -(200 \text{ GPa})(12.5 \times 10^{-6} / C^0)(\Delta T) = -2.5 \Delta T \text{ MPa}$$

Coefficient of Thermal Expansion

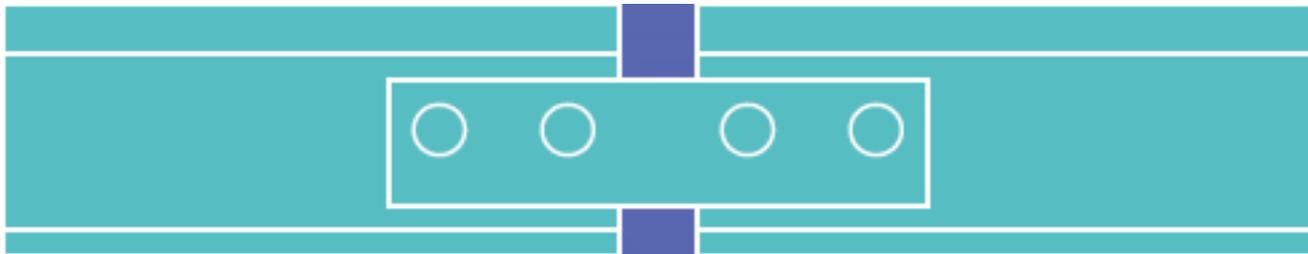
Material	Coefficient of thermal expansion α		Material	Coefficient of thermal expansion α	
	$10^{-6}/^{\circ}\text{F}$	$10^{-6}/^{\circ}\text{C}$		$10^{-6}/^{\circ}\text{F}$	$10^{-6}/^{\circ}\text{C}$
Aluminum alloys	13	23	Plastics		
Brass	10.6–11.8	19.1–21.2	Nylon	40–80	70–140
Bronze	9.9–11.6	18–21	Polyethylene	80–160	140–290
Cast iron	5.5–6.6	9.9–12	Rock	3–5	5–9
Concrete	4–8	7–14	Rubber	70–110	130–200
Copper and copper alloys	9.2–9.8	16.6–17.6	Steel	5.5–9.9	10–18
Glass	3–6	5–11	High-strength	8.0	14
Magnesium alloys	14.5–16.0	26.1–28.8	Stainless	9.6	17
Monel (67% Ni, 30% Cu)	7.7	14	Structural	6.5	12
Nickel	7.2	13	Titanium alloys	4.5–6.0	8.1–11
			Tungsten	2.4	4.3

Addressing Thermo Stresses

- Extension / contraction knots:

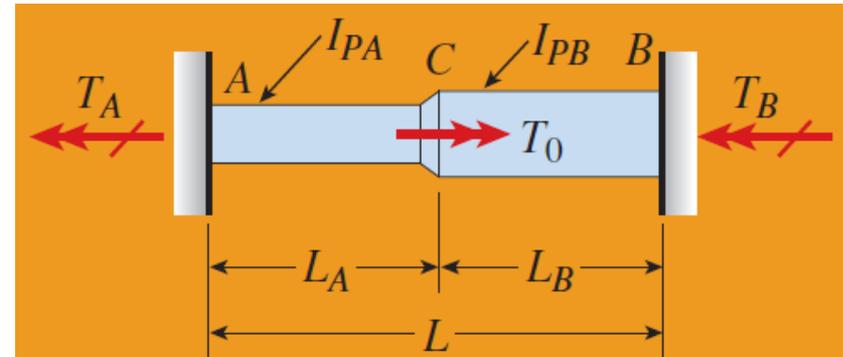
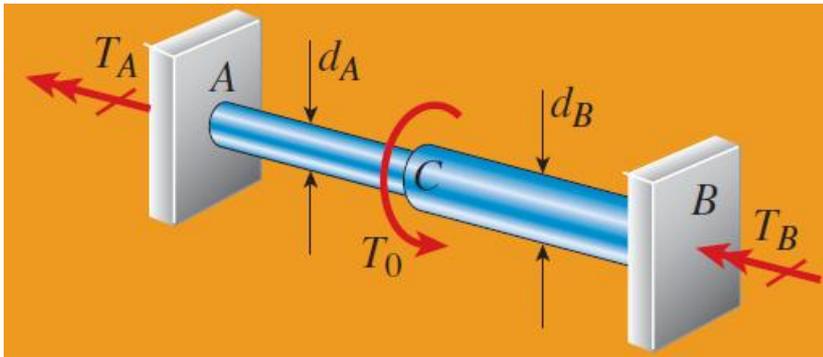


- Extension / contraction slot:



Statically Indeterminate Shafts

- Determine (a) the reactive torques at the ends, (b) the maximum shearing stresses in each segment of the bar, and (c) the angle of rotation at the cross section where the load T_0 is applied.



- Solution:**

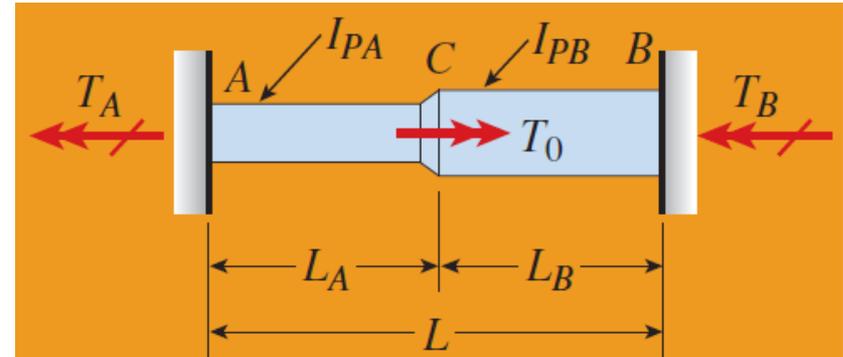
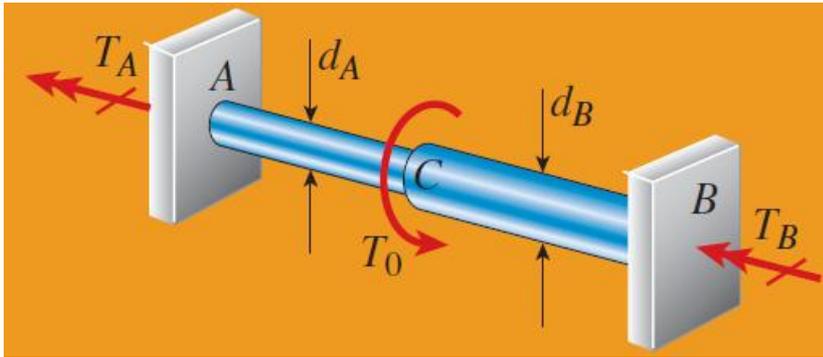
$$\phi_{B/A} = -\frac{T_B L_B}{G I_{pB}} + \frac{(T_0 - T_B) L_A}{G I_{pA}} = 0$$

$$\Rightarrow T_B = \frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} T_0, \quad T_A = \frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} T_0$$

$$\Rightarrow (\tau_{BC})_{\max} = \frac{T_B (d_B/2)}{I_{pB}}, \quad (\tau_{AC})_{\max} = \frac{T_A (d_A/2)}{I_{pA}} \quad \Rightarrow \phi_C = \frac{T_B L_B}{G I_{pB}} = \frac{T_A L_A}{G I_{pA}}$$

Statically Indeterminate Shafts

- For the special case of $d_A = d_B$:



$$\Rightarrow T_B = \frac{L_A I_{pB}}{L_B I_{pA} + L_A I_{pB}} T_0 = \frac{L_A}{L} T_0,$$

$$T_A = \frac{L_B I_{pA}}{L_B I_{pA} + L_A I_{pB}} T_0 = \frac{L_B}{L} T_0$$

$$\Rightarrow (\tau_{BC})_{\max} = \frac{T_B (d/2)}{I_p} = \frac{L_A d T_0}{2 L I_p},$$

$$(\tau_{AC})_{\max} = \frac{T_A (d/2)}{I_p} = \frac{L_B d T_0}{2 L I_p}$$

$$\Rightarrow \phi_C = \frac{T_B L_B}{G I_p} = \frac{T_A L_A}{G I_p} = \frac{L_A L_B T_0}{L G I_p}$$

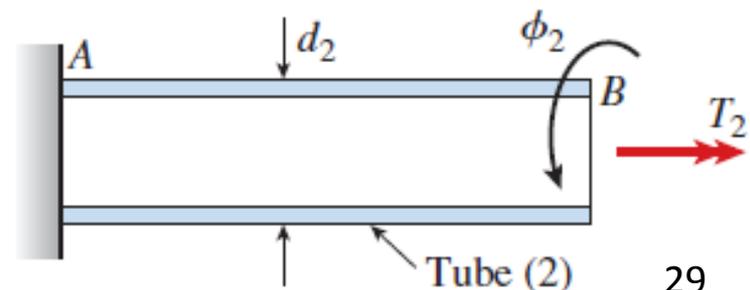
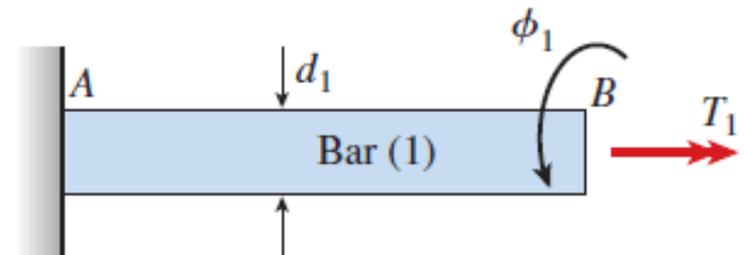
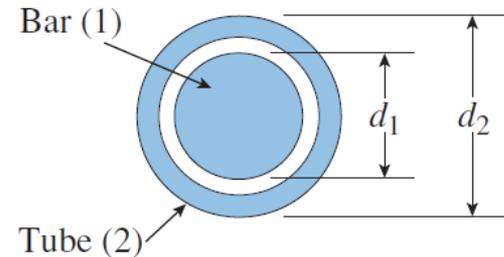
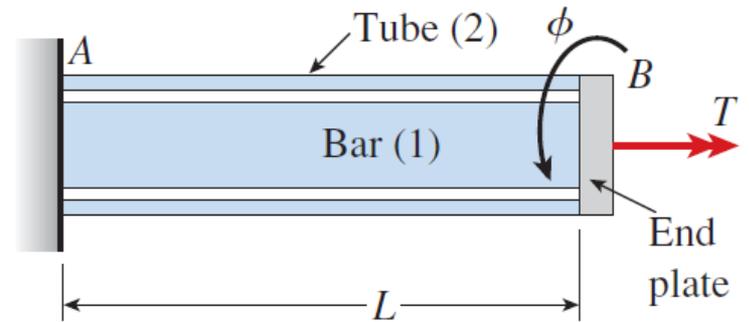
Sample Problem

- What is the deformation of the rod and tube when a torque T is exerted on a rigid end plate as shown?
- Solution:

$$\begin{cases} T_1 + T_2 = T \\ \phi_1 = \frac{T_1 L}{G_1 I_{p1}} = \frac{T_2 L}{G_2 I_{p2}} = \phi_2 \end{cases}$$

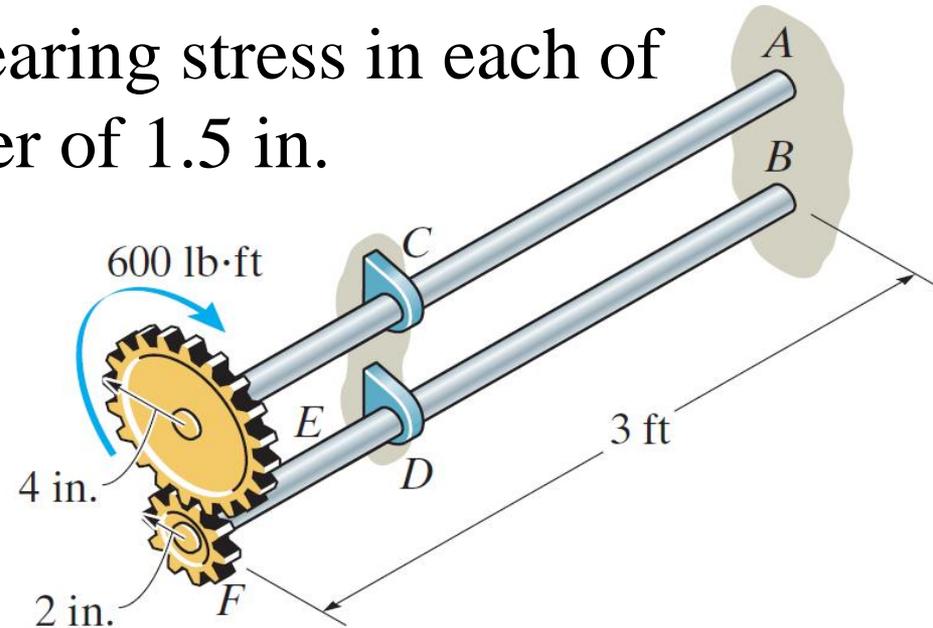
$$\Rightarrow T_1 = \frac{G_1 I_{p1} T}{G_1 I_{p1} + G_2 I_{p2}}, \quad T_2 = \frac{G_2 I_{p2} T}{G_1 I_{p1} + G_2 I_{p2}}$$

$$\Rightarrow \phi_1 = \phi_2 = \frac{TL}{G_1 I_{p1} + G_2 I_{p2}}$$



Sample Problem

- Determine the maximum shearing stress in each of the two shafts with a diameter of 1.5 in.
- Solution:



$$\begin{cases} T_A + F(4/12) = 600 \\ T_B - F(2/12) = 0 \\ 2\phi_F = 2\frac{T_B L}{GI_p} = 4\phi_E = 4\frac{T_A L}{GI_p} \end{cases}$$

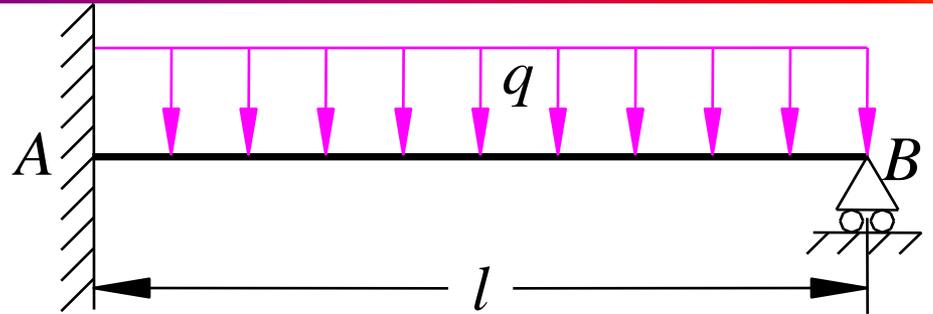
$$\Rightarrow T_A = 120 \text{ lb}\cdot\text{ft}, \quad T_B = 240 \text{ lb}\cdot\text{ft}, \quad F = 1440 \text{ lb}$$

$$\Rightarrow (\tau_{BD})_{\max} = \frac{T_B (d/2)}{I_p} = \frac{240(12)(1.5/2)}{\pi 1.5^4/32} = 4.35 \text{ ksi}$$

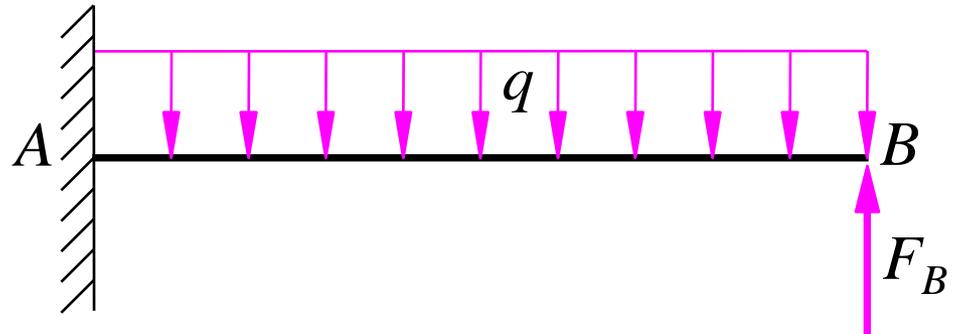
$$\Rightarrow (\tau_{AC})_{\max} = \frac{T_A (d/2)}{I_p} = \frac{1}{2} (\tau_{BD})_{\max} = 2.17 \text{ ksi}$$

Statically Indeterminate Beams

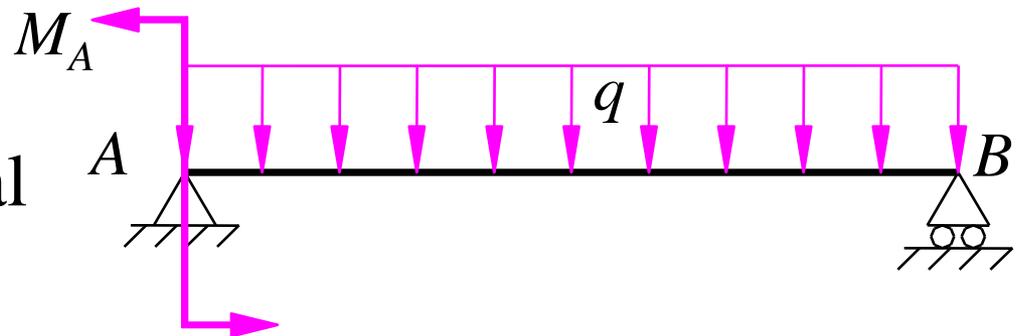
- A propped cantilever beam



- Make the beam determinate by replacing the restraint at B with F_B

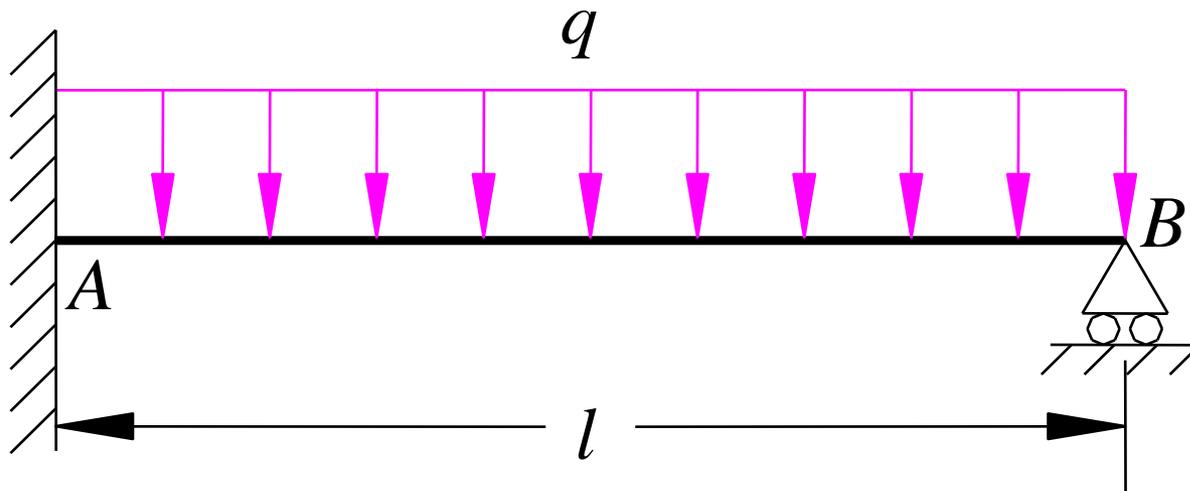


- Make the beam determinate by replacing the rotational restraint at A with M_A



Sample Problem

- Known: q and l . Find: reaction forces at the supports.

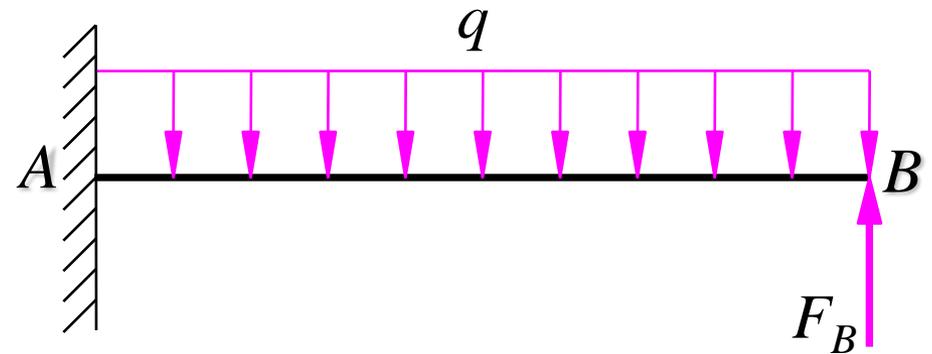
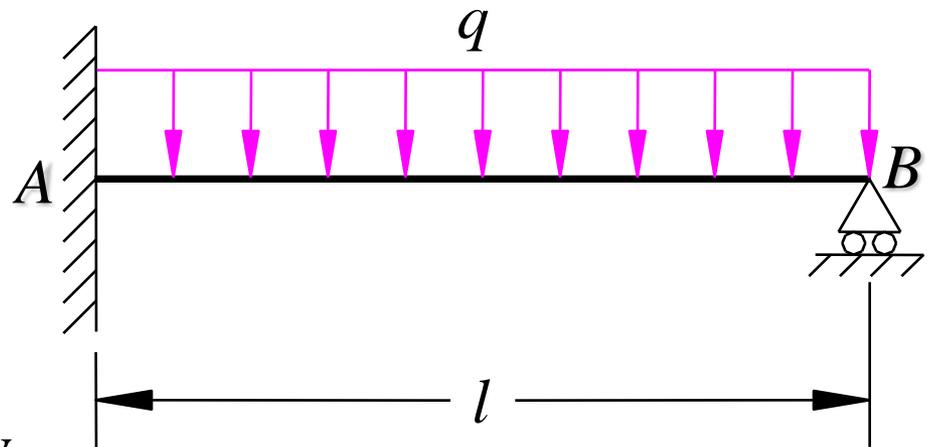


- Method 1:
- Take the deflection restraint at B as the redundant restraint
- Deformation compatibility requires:

$$w_B = 0$$

$$\frac{F_B l^3}{3EI} - \frac{ql^4}{8EI} = 0$$

$$\Rightarrow F_B = \frac{3ql}{8}$$

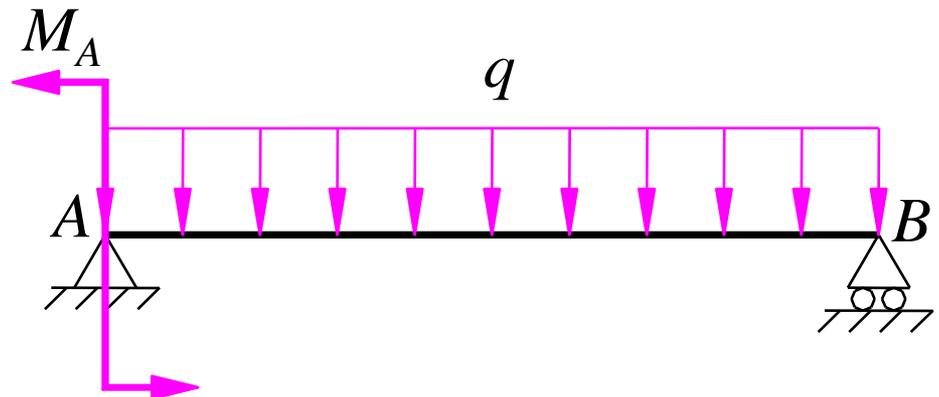
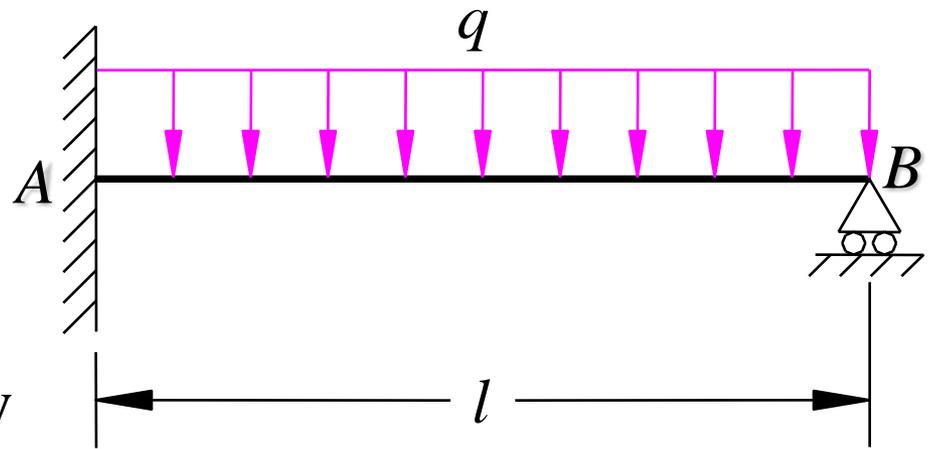


- Method 2:
- Take the rotational restraint at A as the redundant restraint
- Deformation compatibility requires:

$$\theta_A = 0$$

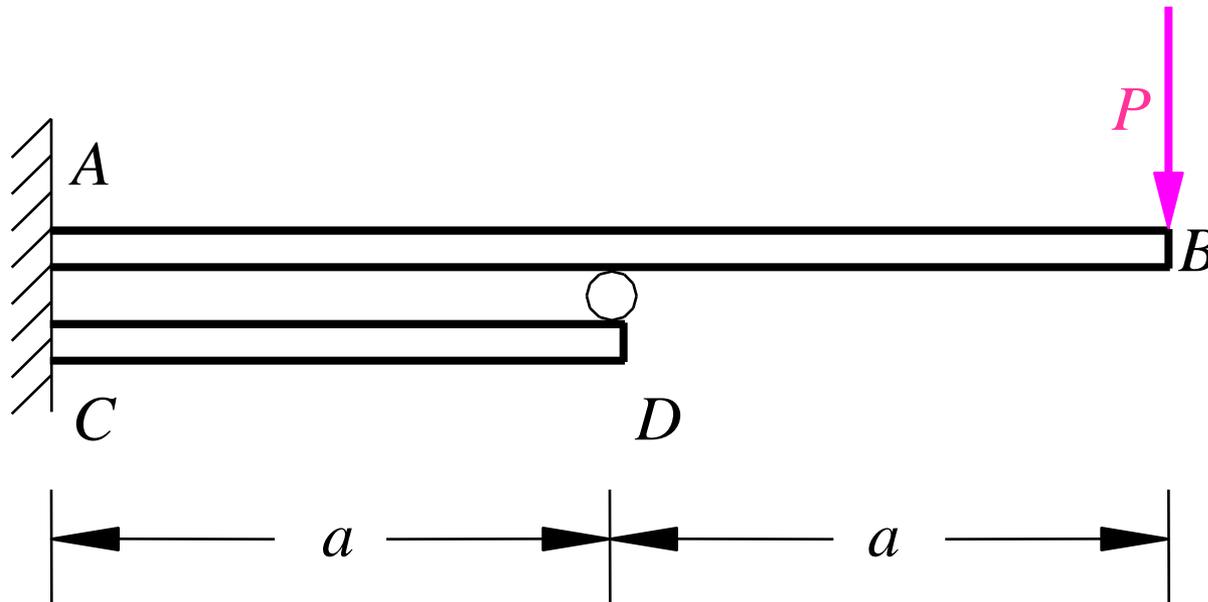
$$\frac{M_A l}{3EI} - \frac{ql^3}{24EI} = 0$$

$$\Rightarrow M_A = \frac{ql^2}{8}$$

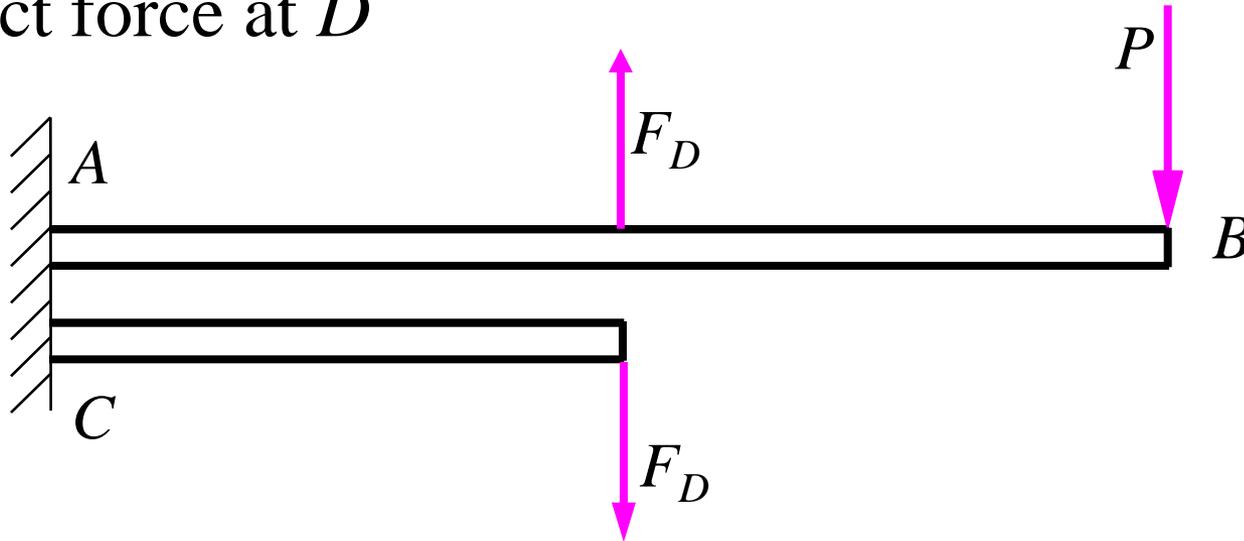


Sample Problem

- Cantilever beam AB is enhanced by another cantilever beam CD with the same EI . Find (1) contact force at D ; (2) ratio of deflections at B with and without enhancement; (3) ratio of the maximum bending moments in AB with and without enhancement.



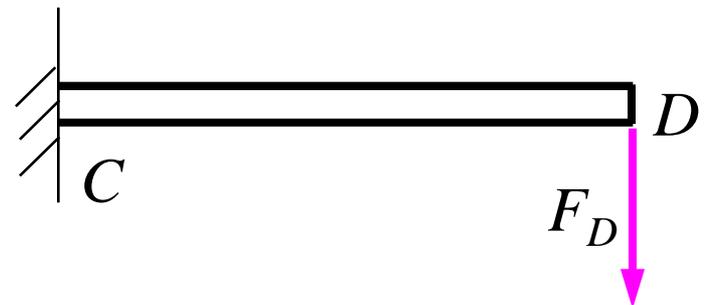
- Solution:
- Contact force at D

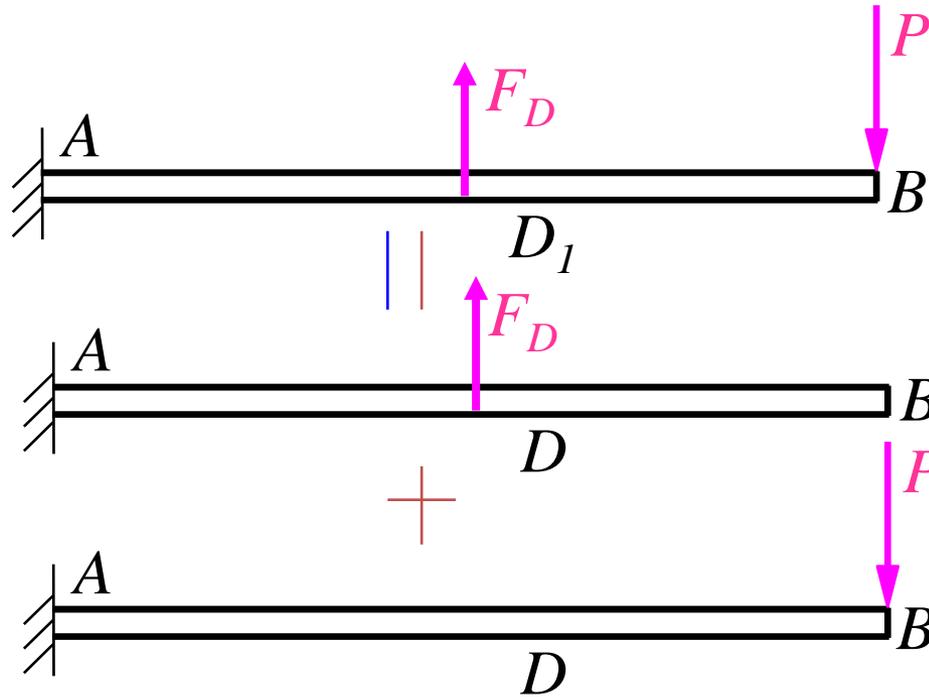


- Deflection compatibility at D requires

$$w_D (AB) = w_D (CD)$$

$$w_D (CD) = \frac{F_D a^3}{3EI}$$





$$w_D(AB) = -\frac{F_D a^2}{6EI} (3a - a) + \frac{Pa^2}{6EI} (3(2a) - a) = -\frac{F_D a^3}{3EI} + \frac{5Pa^3}{6EI}$$

$$w_D(AB) = w_D(CD) \Rightarrow -\frac{F_D a^3}{3EI} + \frac{5Pa^3}{6EI} = \frac{F_D a^3}{3EI} \Rightarrow F_D = \frac{5}{4}P$$

- Ratio of deflections at B with and without the enhancement

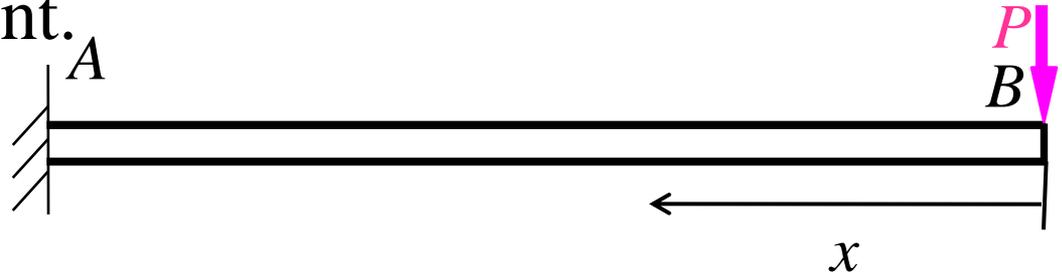
$$w_{B0} = \frac{P(2a)^3}{3EI} = \frac{8Pa^3}{3EI}$$

$$w_{B1} = -\frac{F_D a^2}{6EI} (3(2a) - a) + \frac{8Pa^3}{3EI} = -\frac{5F_D a^3}{6EI} + \frac{8Pa^3}{3EI} = \frac{39Pa^3}{24EI}$$

$$\Rightarrow \frac{w_{B1}}{w_{B0}} = \frac{39Pa^3}{24EI} \bigg/ \frac{8Pa^3}{3EI} = \frac{39}{64}$$

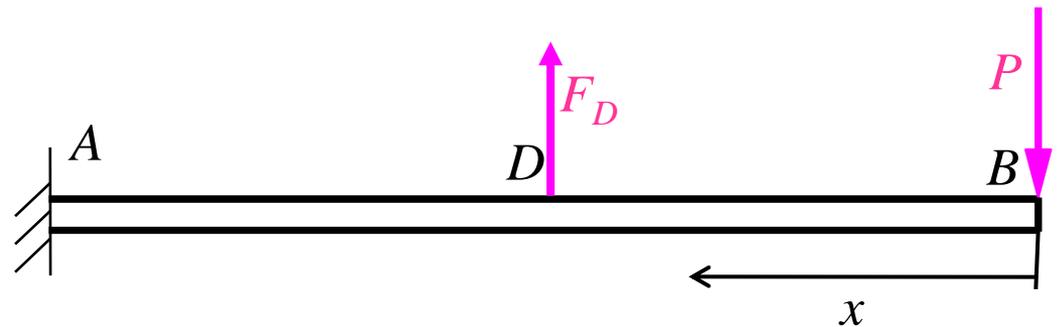
- Ratio of the maximum bending moments in AB with and without the enhancement.

- Without:



$$M_0(x) = -Px \Rightarrow M_{0\max} = M_{0A} = -2Pa$$

- With:

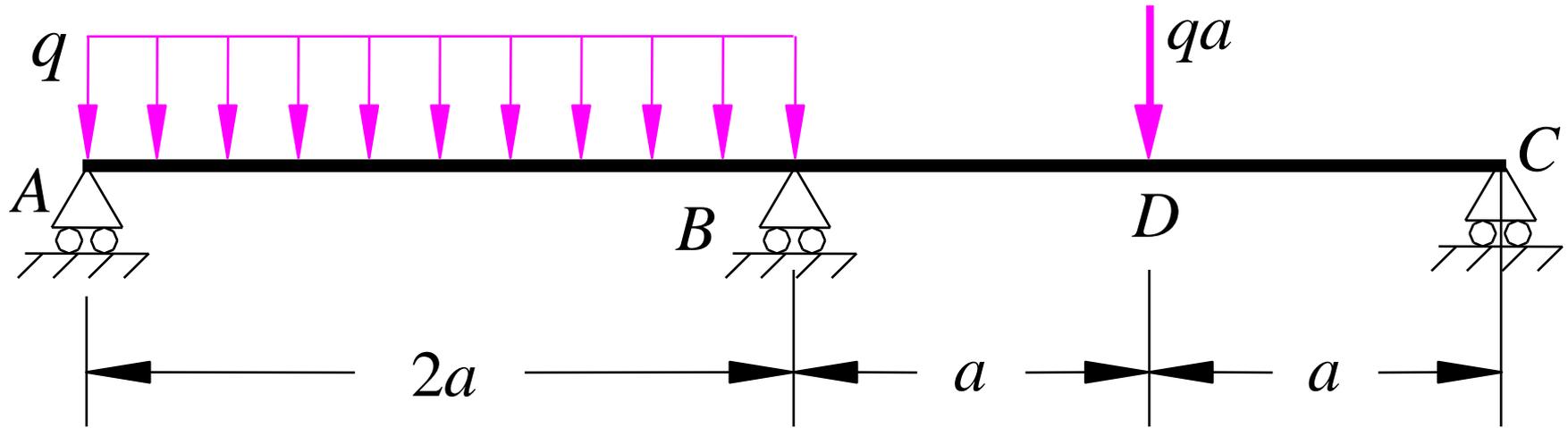


$$M_1(x) = \begin{cases} -Px & x \leq a \\ -Px + F_D(x - a) & a \leq x \leq 2a \end{cases}$$

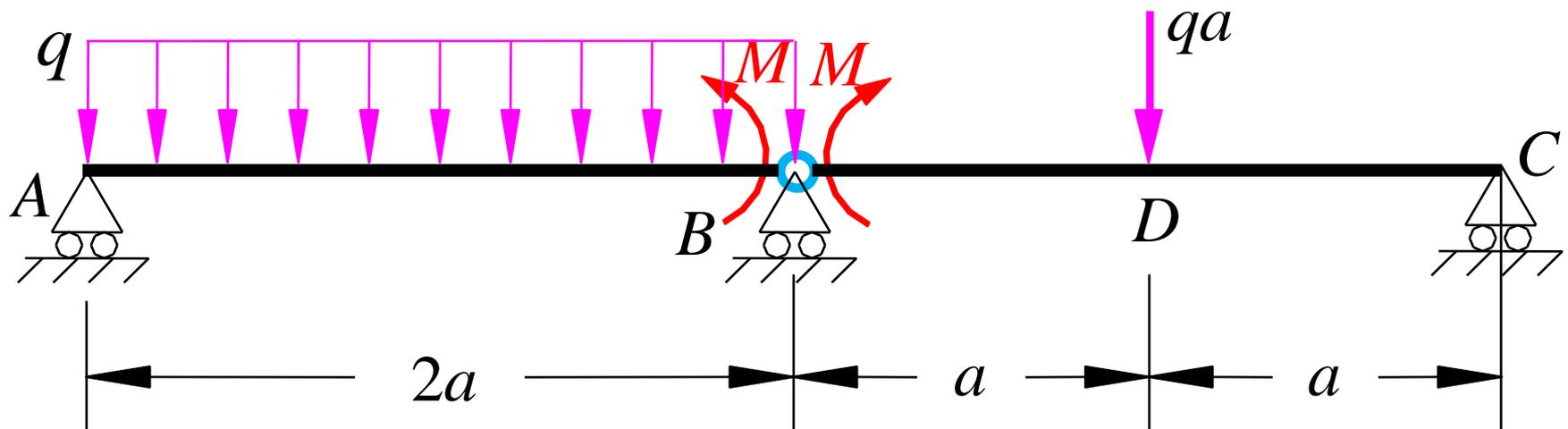
$$\Rightarrow M_{1\max} = M_{1D} = -Pa$$

Exercise

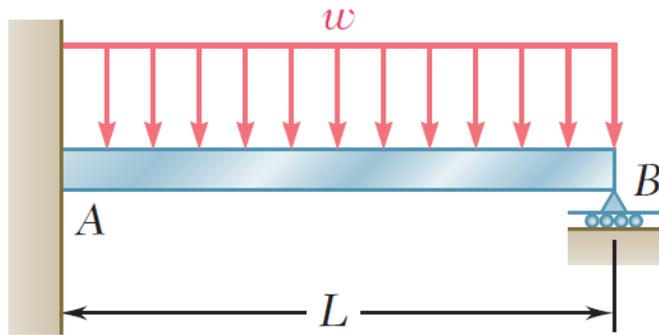
- Find the reaction forces for the following continuous beam.



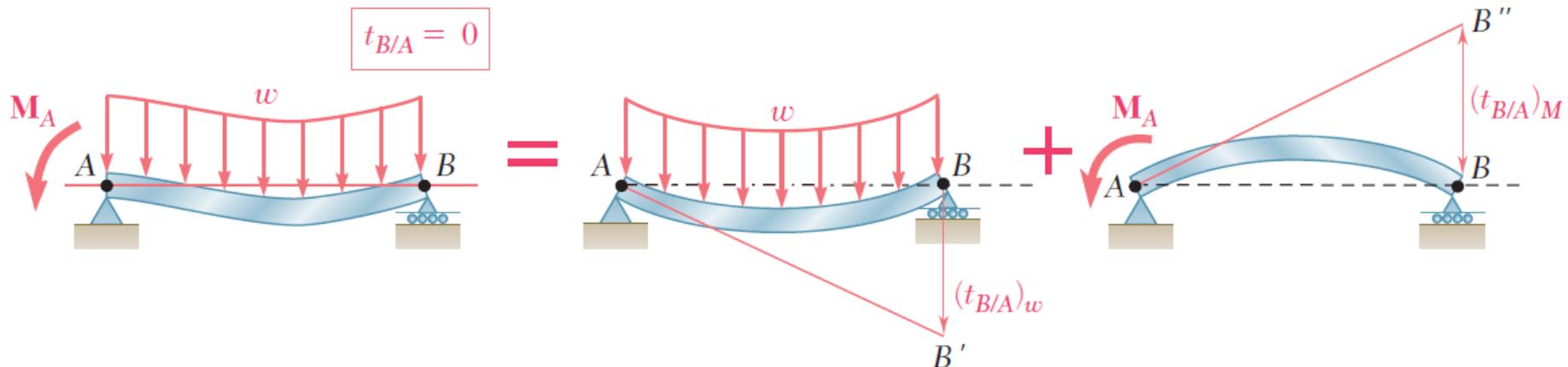
- Solution



Moment-Area Theorems with Indeterminate Beams

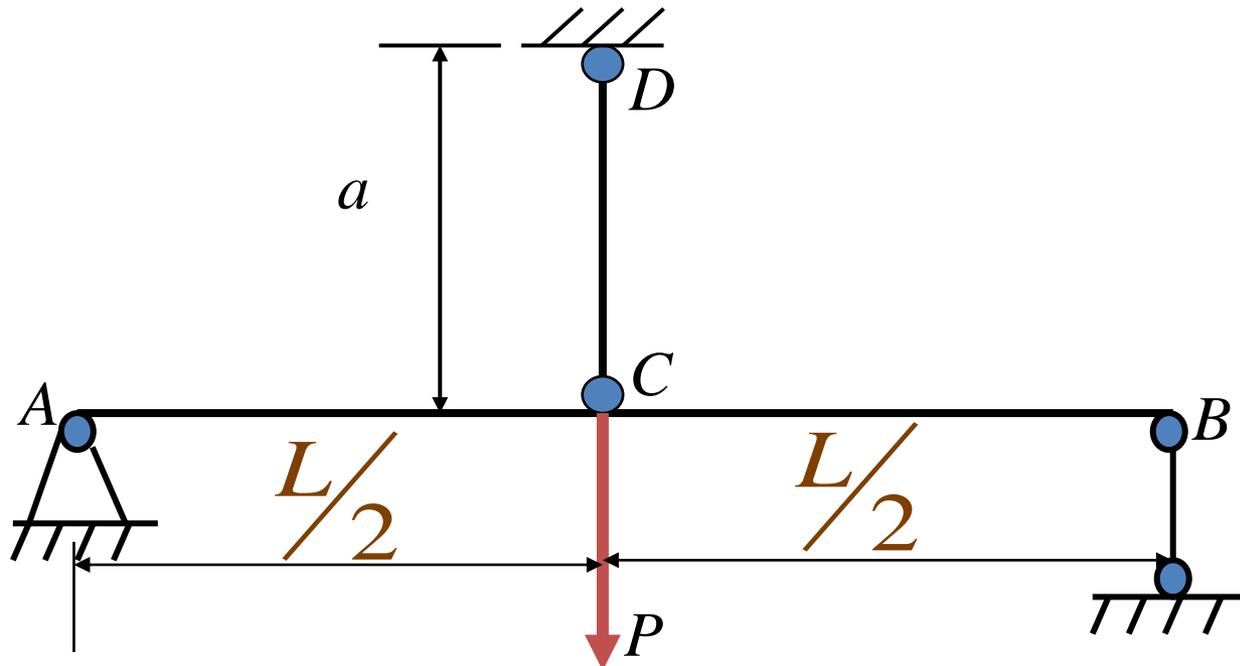


- Reactions at supports of statically indeterminate beams are found by designating a redundant constraint and treating it as an unknown load which satisfies a displacement compatibility requirement.
- The (M/EI) diagram is drawn by parts. The resulting tangential deviations are superposed and related by the compatibility requirement.
- With reactions determined, the slope and deflection are found from the moment-area method.



Combined Indeterminate Structures

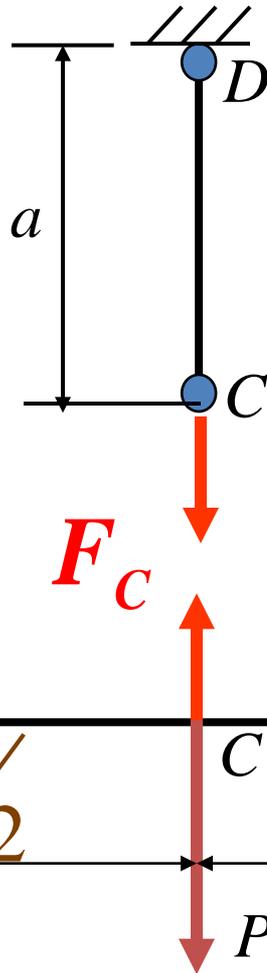
- For the coplanar structure shown below, the flexural rigidity of AB is EI , tension rigidity of CD is EA ; P , L and a are given. Find F_{CD} .



Combined Indeterminate Structures

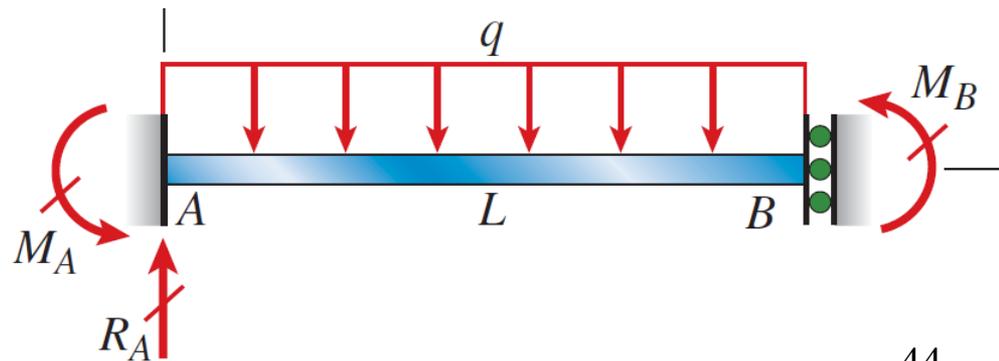
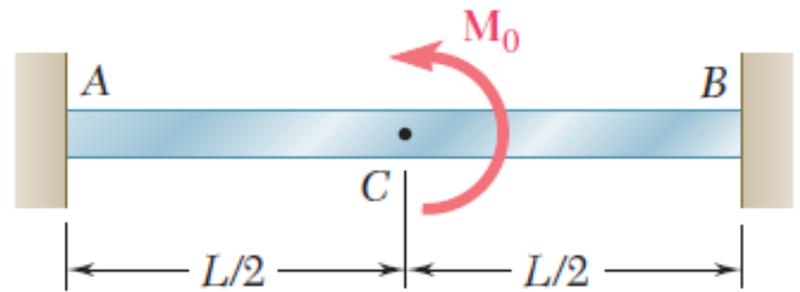
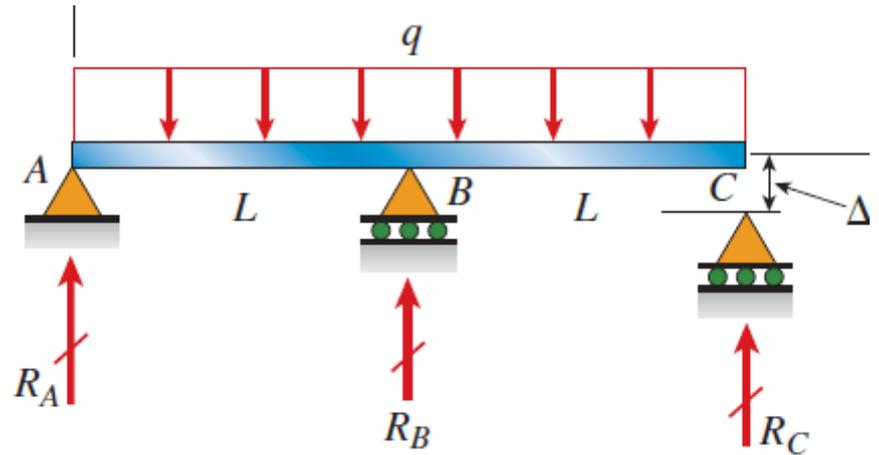
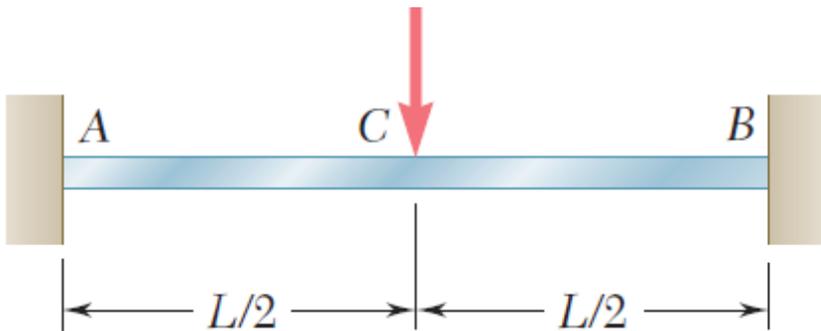
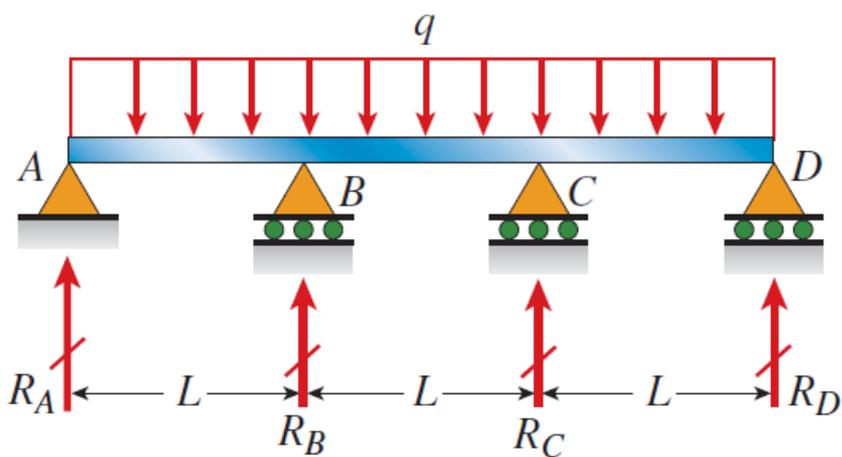
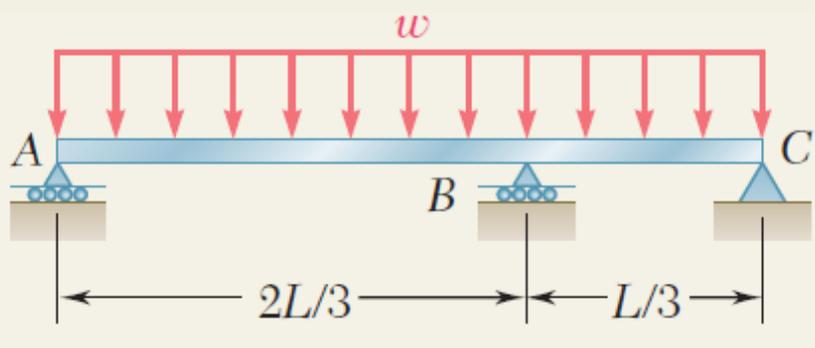
- Solution

- Deformation compatibility requires:

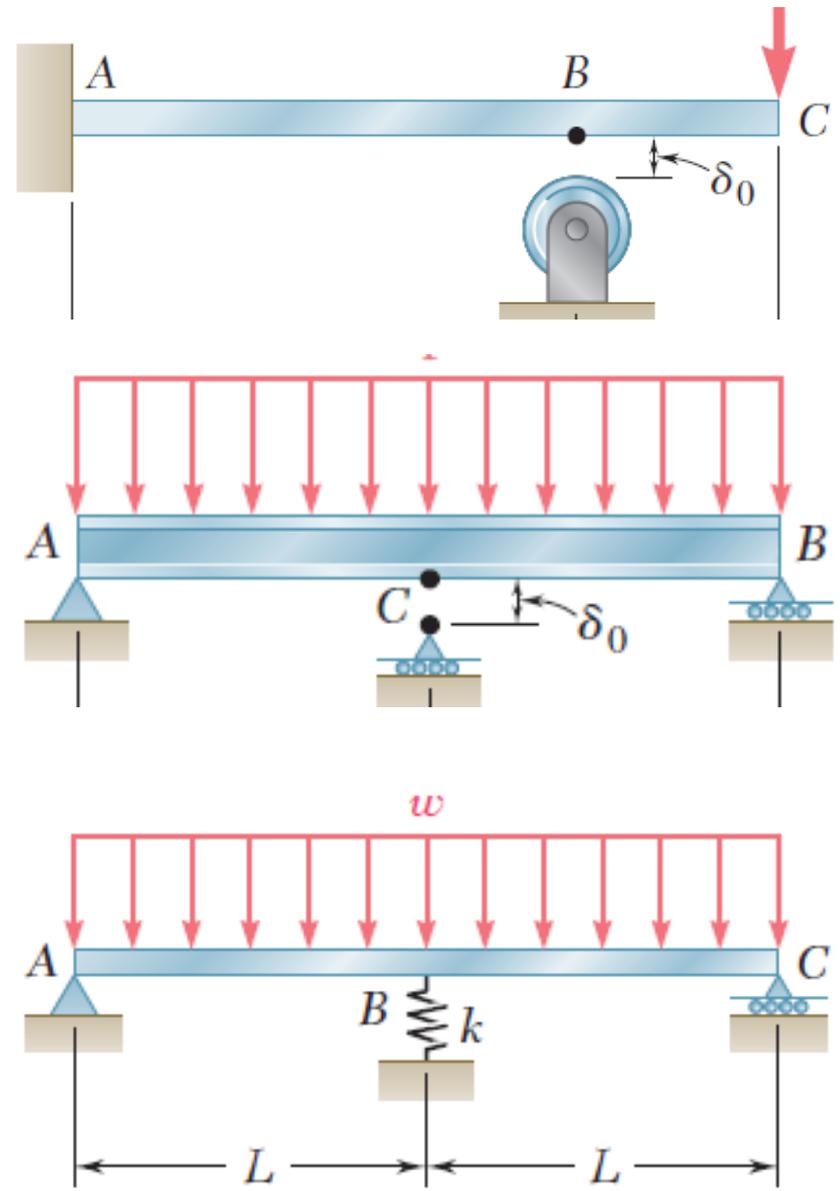
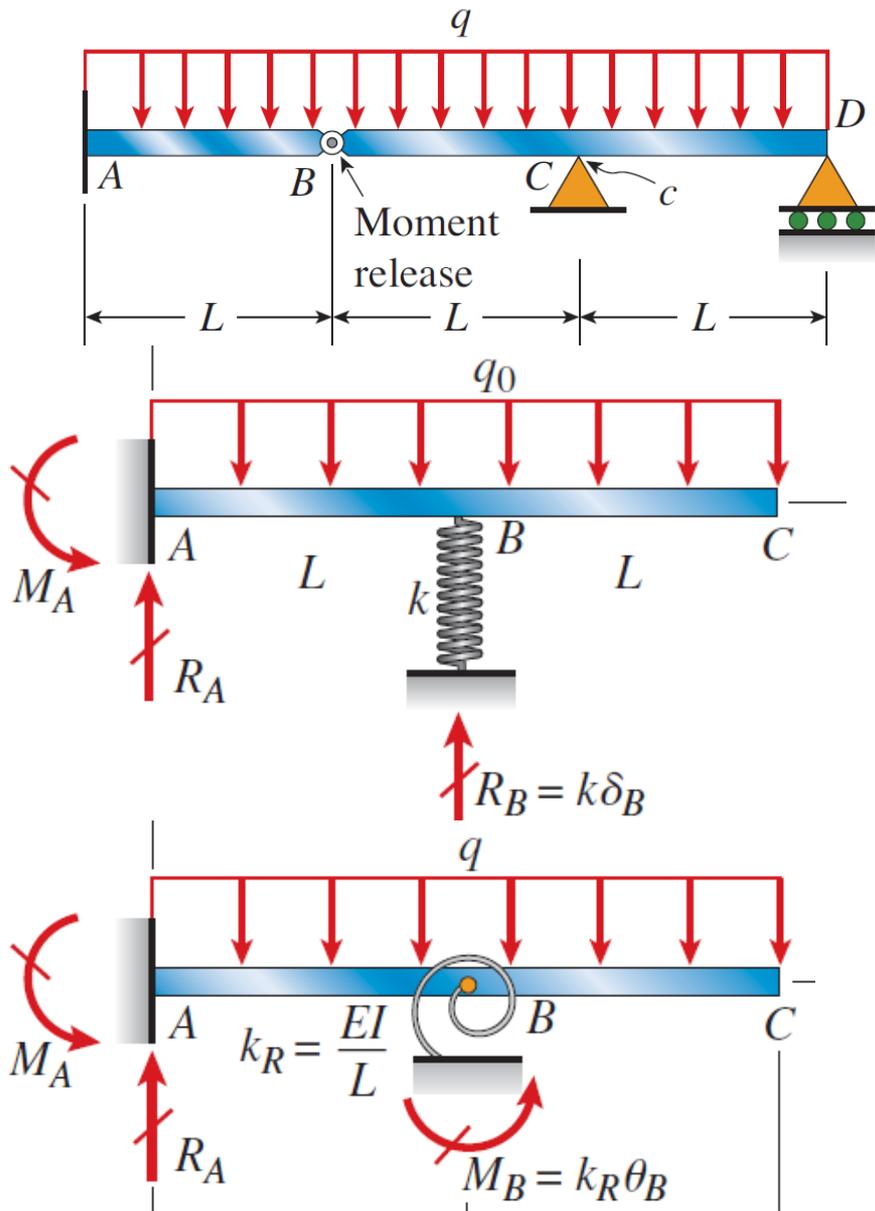


$$w_C = \Delta l_{CD}$$
$$\Rightarrow w_C = \frac{(P - F_C)L^3}{48EI} = \Delta l_{CD} = \frac{F_C a}{EA}$$
$$\Rightarrow F_C = \frac{AL^3}{48La + AL^3} P$$

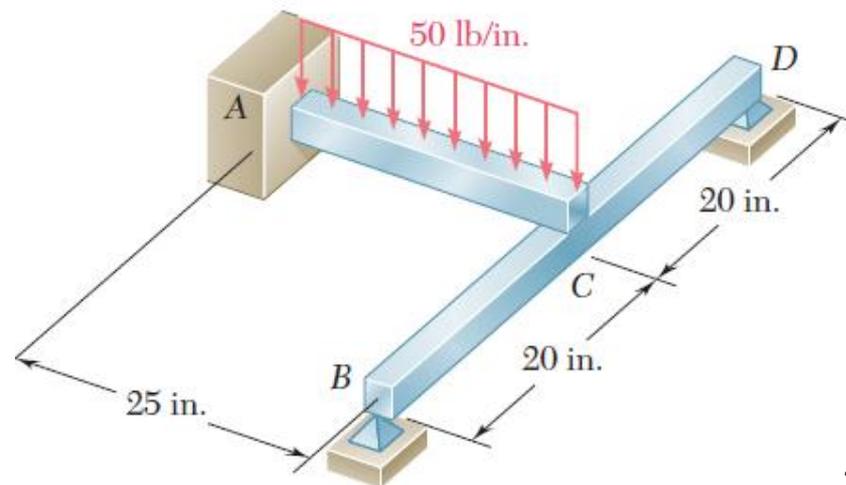
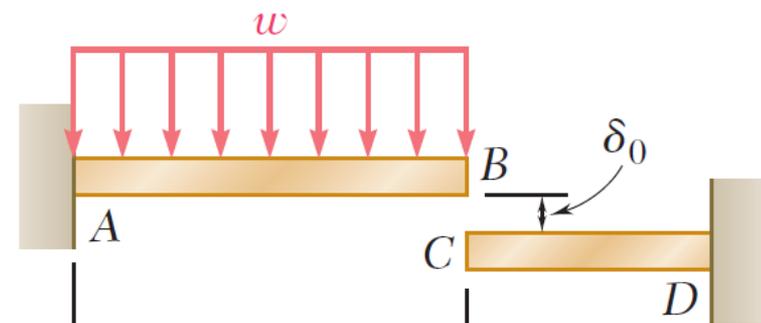
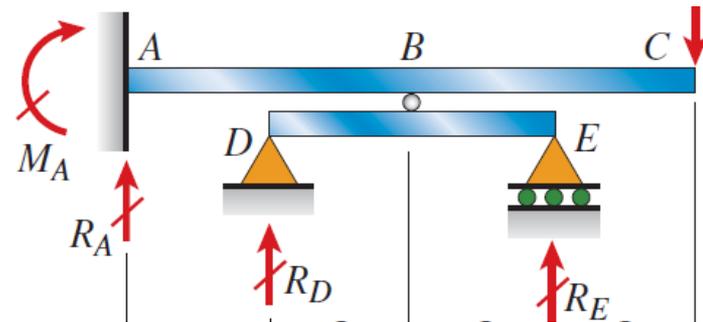
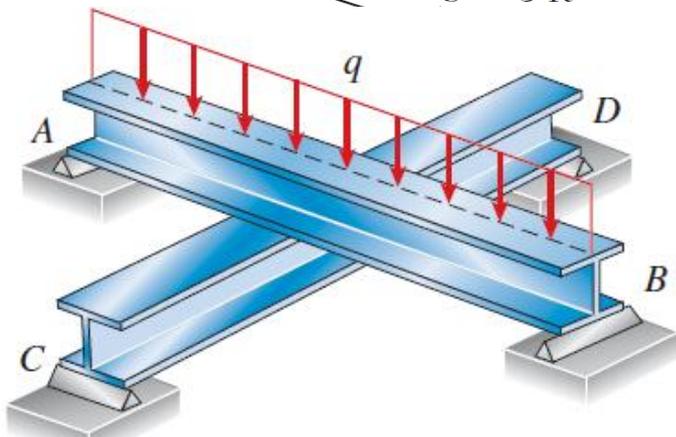
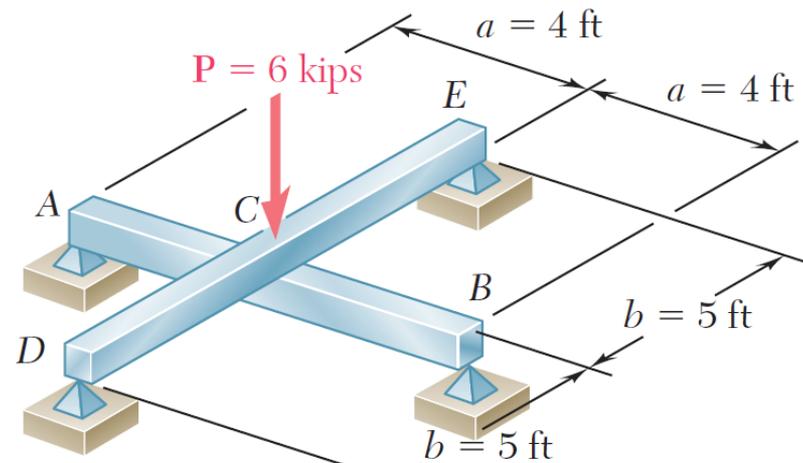
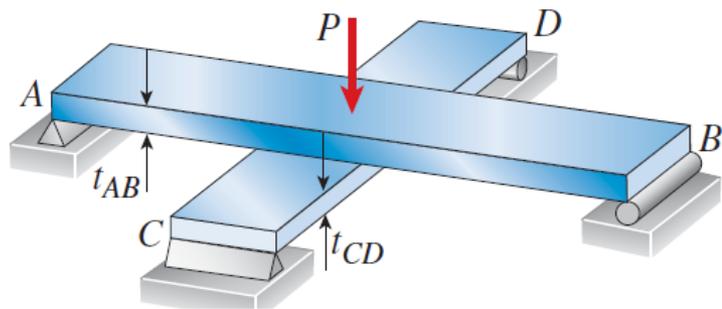
More Examples



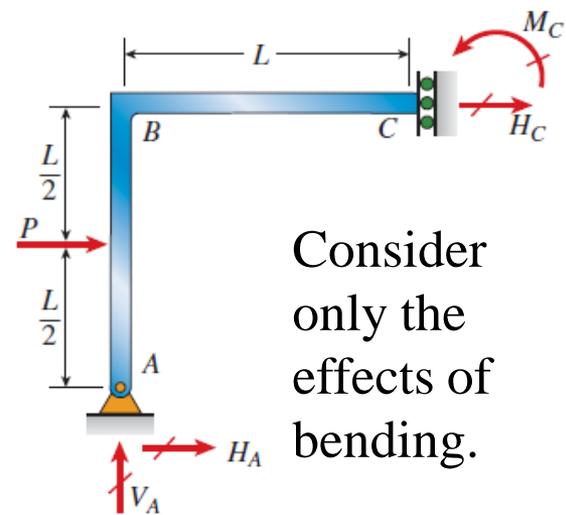
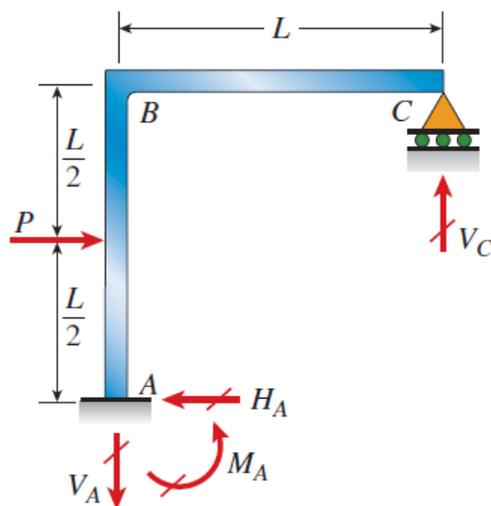
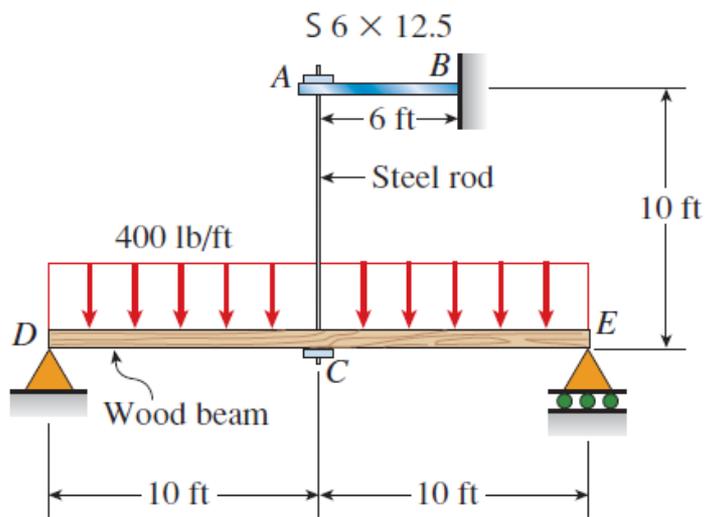
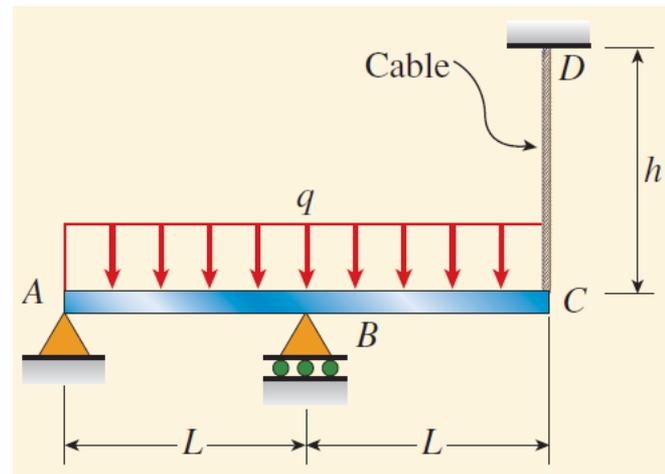
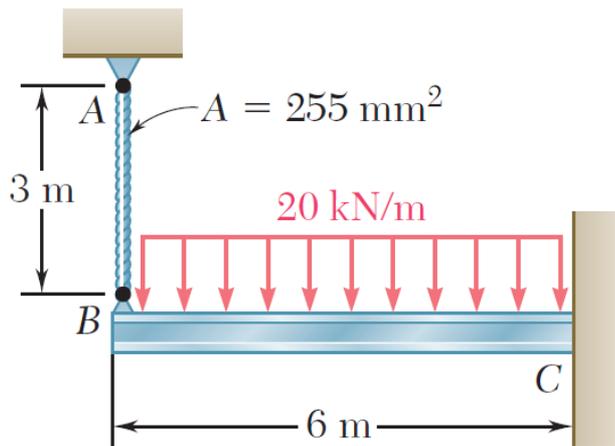
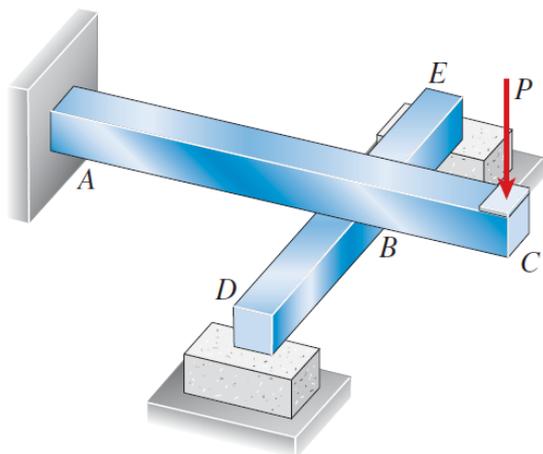
More Examples



More Examples



More Examples



Consider only the effects of bending.

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