Simple Linear Elastic BVPs

mi@seu.edu.cn

Outline

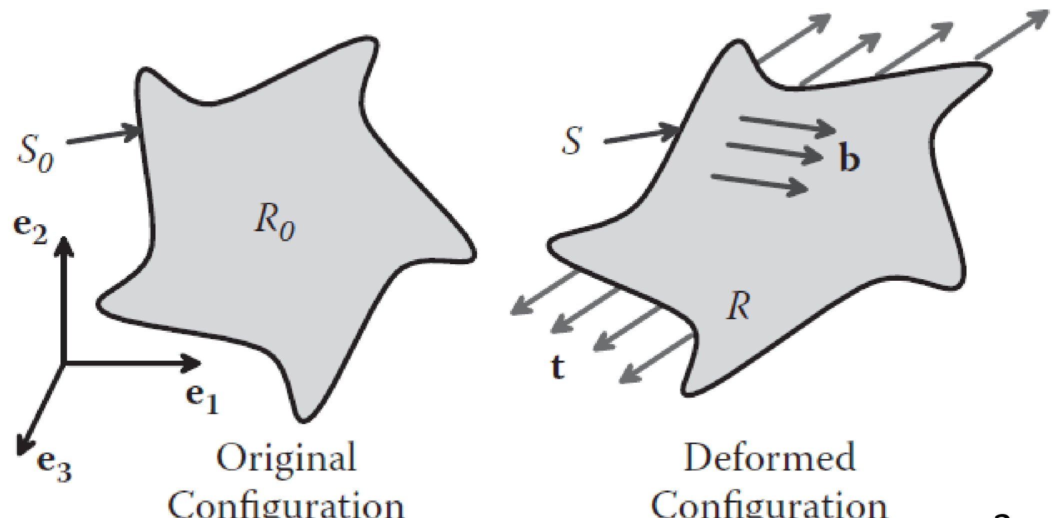
- Review of field equations（线弹性力学控制方程回顾）
- Thermoelasticity（热弹性力学本构关系）
- Small strain theory in cylindrical coordinates（柱坐标）
- Axial symmetry（轴对称）
- Pressurized cylindrical shell（压力圆筒）
- Spinning disk（圆筒转动）
- Interference fit between two cylinders（圆筒过盈装配）
- Small strain theory in spherical coordinates（球坐标系）
- Spherical symmetry（球对称）
- Pressurized spherical shell（压力球腔）
- Gravitating planet（重力球）
- Steady-state heat flow in spherical shell（球腔稳态热流）

Review of Field Equations

- Strain-displacement relations: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Strain compatibility: $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$
- Equilibrium: $\sigma_{ij,i} + F_j = \sigma_{ij,i} + \rho b_j = 0.$
- Isotropic Hooke's Law:

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right\}; \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}.$$

- Traction BCs on S_t
- Displacement BCs on S_u



Thermoelastic Constitutive Relations

- A temperature change in an elastic solid produces deformation.
- The total strain can be decomposed into the sum of mechanical and thermal components.
- It is extremely important to understand that, the elastic stiffness tensor (***C***) correlates mechanical stress and mechanical strain.

$$\varepsilon_{ij}^{\text{Total}} = \varepsilon_{ij}^{M} + \varepsilon_{ij}^{T} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}^{M}\delta_{ij} + 2G\varepsilon_{ij}^{M} = \lambda\left(\varepsilon_{kk}^{\text{Total}} - \varepsilon_{kk}^{T}\right)\delta_{ij} + 2G\left(\varepsilon_{ij}^{\text{Total}} - \varepsilon_{ij}^{T}\right)$$

$$\sigma_{ij} = \lambda\varepsilon_{kk}^{\text{Total}}\delta_{ij} + 2G\varepsilon_{ij}^{\text{Total}} - (3\lambda + 2G)\alpha\Delta T\delta_{ij} = \lambda\varepsilon_{kk}^{\text{Total}}\delta_{ij} + 2G\varepsilon_{ij}^{\text{Total}} - 3K\alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\varepsilon_{kk}^{\text{Total}}\delta_{ij} + \varepsilon_{ij}^{\text{Total}}\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij}$$

Cylindrical Strain and Rotation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla} + \nabla\mathbf{u}\right); \quad \boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla} - \nabla\mathbf{u}\right); \ \mathbf{u} = u_r\mathbf{e_r} + u_\theta\mathbf{e_\theta} + u_z\mathbf{e_z};$$

$$\mathbf{u}\overleftarrow{\nabla}_c = \begin{bmatrix} \frac{\partial u_r}{\partial r}\mathbf{e}_r\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right)\mathbf{e}_r\mathbf{e}_\theta + \frac{\partial u_r}{\partial z}\mathbf{e}_r\mathbf{e}_z + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\mathbf{e}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right)\mathbf{e}_\theta\mathbf{e}_\theta \\ + \frac{\partial u_\theta}{\partial z}\mathbf{e}_\theta\mathbf{e}_z + \frac{\partial u_z}{\partial r}\mathbf{e}_z\mathbf{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\mathbf{e}_z\mathbf{e}_\theta + \frac{\partial u_z}{\partial z}\mathbf{e}_z\mathbf{e}_z \end{bmatrix}$$

$$\omega_r = \omega_\theta = \omega_z = 0, \omega_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} - \frac{\partial u_\theta}{\partial r}\right),$$

$$\omega_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} - \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right), \omega_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}\right);$$

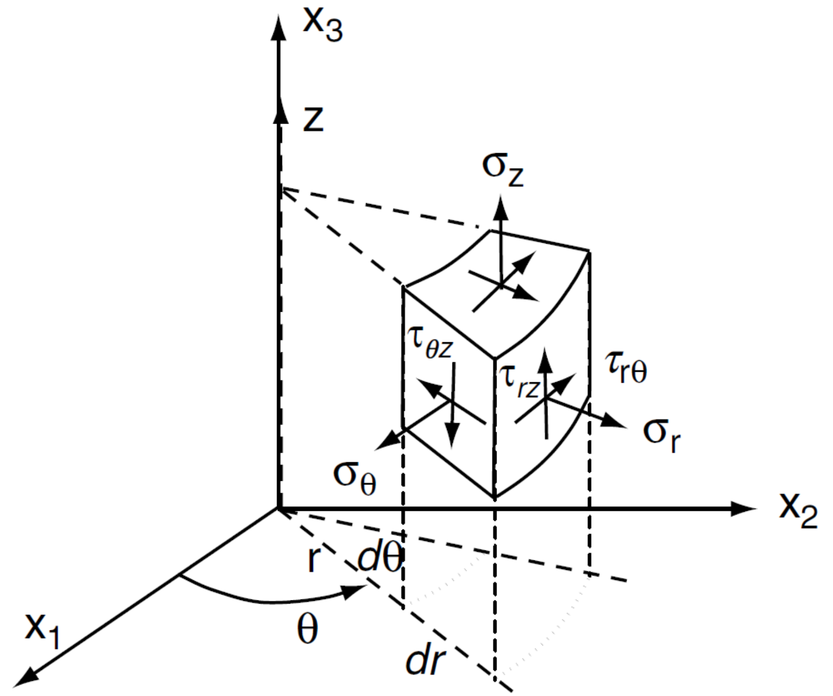
$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \varepsilon_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial\theta}\right), \varepsilon_z = \frac{\partial u_z}{\partial z}, \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right),$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right), \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right).$$

Cylindrical Equilibrium Equations

$\nabla \cdot \boldsymbol{\sigma}$ = contraction on the first and third index of $\boldsymbol{\sigma}\overleftarrow{\nabla}$

$$\Rightarrow \boxed{\begin{aligned}\nabla \cdot \boldsymbol{\sigma} = &\left(\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r}\right)\mathbf{e}_r \\ &+ \left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r}\right)\mathbf{e}_\theta \\ &+ \left(\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r}\right)\mathbf{e}_z\end{aligned}}$$



$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{F} = \mathbf{0}$$

$$\Rightarrow \boxed{\begin{aligned}&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0, \\ &\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0, \\ &\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + F_z = 0.\end{aligned}}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$$

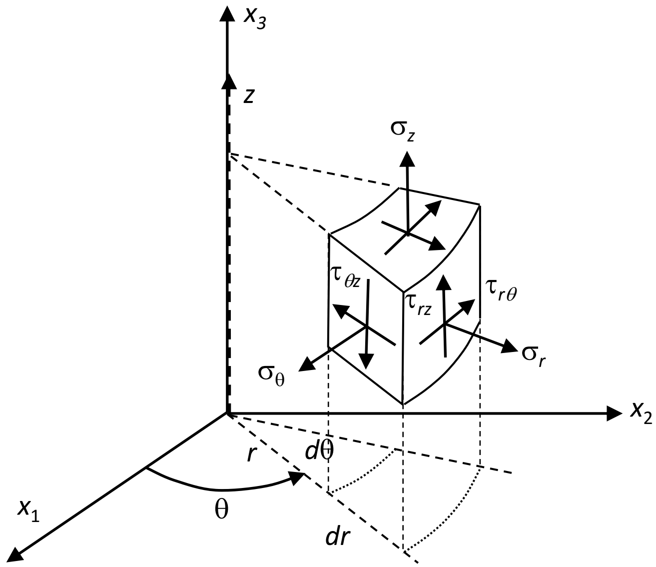
$$\mathbf{T}^{\mathbf{r}} = \sigma_r \mathbf{e}_r + \tau_{r\theta}\mathbf{e}_\theta + \tau_{rz}\mathbf{e}_z$$

$$\mathbf{T}^{\boldsymbol{\theta}} = \tau_{r\theta}\mathbf{e}_r + \sigma_\theta \mathbf{e}_\theta + \tau_{\theta z}\mathbf{e}_z$$

$$\mathbf{T}^{\mathbf{z}} = \tau_{rz}\mathbf{e}_r + \tau_{\theta z}\mathbf{e}_\theta + \sigma_z \mathbf{e}_z$$

Hooke’s Law in Cylindrical Coordinates

- Recall that, **the elastic stiffness tensor *C* is a fourth order isotropic tensor**.
- Its components remain unchanged under any orthogonal coordinate systems.
- The isotropic Hooke’s law stays the same.



$$\varepsilon_{ij}^{\text{Total}} = \varepsilon_{ij}^{M} + \varepsilon_{ij}^{T} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij},$$

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij}.$$

$$\sigma_r = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + \varepsilon_r\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_\theta = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + \varepsilon_\theta\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_z = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + \varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\tau_{r\theta} = \frac{E}{(1+\nu)}\varepsilon_{r\theta}, \tau_{z\theta} = \frac{E}{(1+\nu)}\varepsilon_{z\theta}, \tau_{rz} = \frac{E}{(1+\nu)}\varepsilon_{rz}.$$

Axial Symmetry

- Displacements and stresses

$$\mathbf{u} = u_r[r]\mathbf{e}_r + \varepsilon_z z \mathbf{e}_z, \quad \boldsymbol{\sigma} = \sigma_r[r]\mathbf{e}_r\mathbf{e}_r + \sigma_\theta[r]\mathbf{e}_\theta\mathbf{e}_\theta + \sigma_z[r]\mathbf{e}_z\mathbf{e}_z$$

- Strain-displacement relation:

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u_r}{r}$$

- Equations of motion:

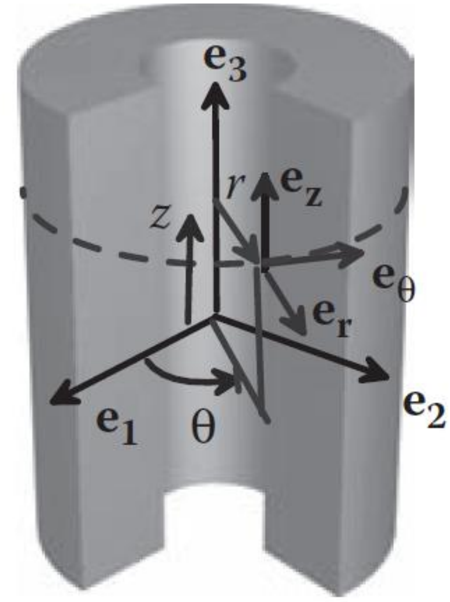
$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = -\rho\omega^2 r$$

- Hooke's law in cylindrical coordinates

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\varepsilon_r + \nu\varepsilon_\theta + \nu\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\varepsilon_r + (1-\nu)\varepsilon_\theta + \nu\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\varepsilon_r + \nu\varepsilon_\theta + (1-\nu)\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}.$$



Plane strain, or
generalized plane strain

- Plane strain

$$\varepsilon_z = 0.$$

- Generalized plane strain

$$\varepsilon_z = \text{const.}, \quad F_z = \int_a^b 2\pi r \sigma_z dr.$$

Axial Symmetry

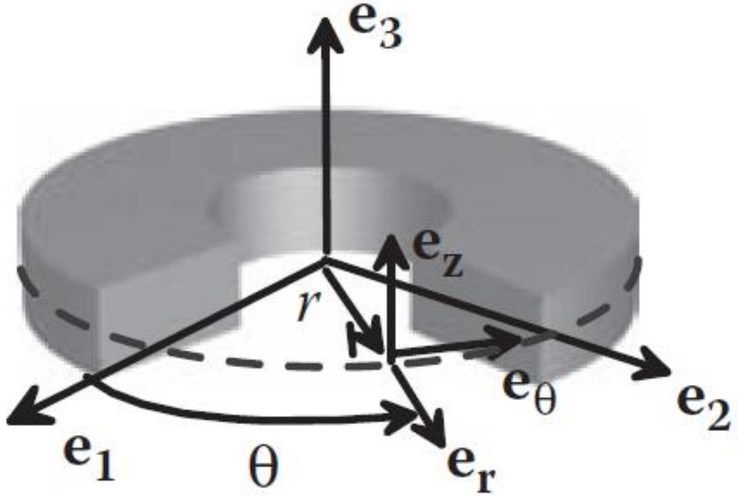
- Plane stress

$$\varepsilon_{ij}^{\text{Total}} = \varepsilon_{ij}^{M} + \varepsilon_{ij}^{T} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}, \quad \sigma_{ij} = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij}.$$

$$\Rightarrow 0 = \sigma_z = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\theta) + \frac{1-\nu}{1-2\nu}\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)} \quad \Rightarrow \varepsilon_z = \frac{(1+\nu)\alpha\Delta T}{1-\nu} - \frac{\nu}{1-\nu}(\varepsilon_r + \varepsilon_\theta)$$

$$\Rightarrow \varepsilon_{kk} = \varepsilon_r + \varepsilon_\theta + \varepsilon_z = \frac{(1+\nu)\alpha\Delta T}{1-\nu} + \frac{1-2\nu}{1-\nu}(\varepsilon_r + \varepsilon_\theta)$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) - \frac{E\alpha\Delta T}{(1-\nu)}, \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\nu\varepsilon_r + \varepsilon_\theta) - \frac{E\alpha\Delta T}{(1-\nu)}, \\ \sigma_z &= 0, \quad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) + \alpha\Delta T. \end{aligned}}$$



Plane stress

- Boundary conditions: $u_r[a] = u_a, \quad u_r[b] = u_b$

$$\sigma_r[a] = \sigma_a, \quad \sigma_r[b] = \sigma_b$$

Axial Symmetry

- Stresses in terms of displacements (generalized plane strain)

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r} + \nu\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\mathrm{d}u_r}{\mathrm{d}r} + (1-\nu)\frac{u_r}{r} + \nu\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r} + (1-\nu)\varepsilon_z\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}.$$

- Stresses in terms of displacements (plane stress)

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r}\right) - \frac{E\alpha\Delta T}{(1-\nu)}, \quad \sigma_\theta = \frac{E}{1-\nu^2}\left(\nu\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r}\right) - \frac{E\alpha\Delta T}{(1-\nu)}.$$

- Equilibrium equations in terms of displacements (generalized plane strain)

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = -F_r - \rho\omega^2 r$$

$$\Rightarrow \frac{\mathrm{d}^2u_r}{\mathrm{d}r^2} - \frac{u_r}{r^2} + \frac{1}{r}\frac{\mathrm{d}u_r}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_r)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}r} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\left(F_r + \rho\omega^2 r\right).$$

Axial Symmetry

- Equilibrium equations in terms of displacements (plane stress)

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=-F_r-\rho\omega^2 r$$

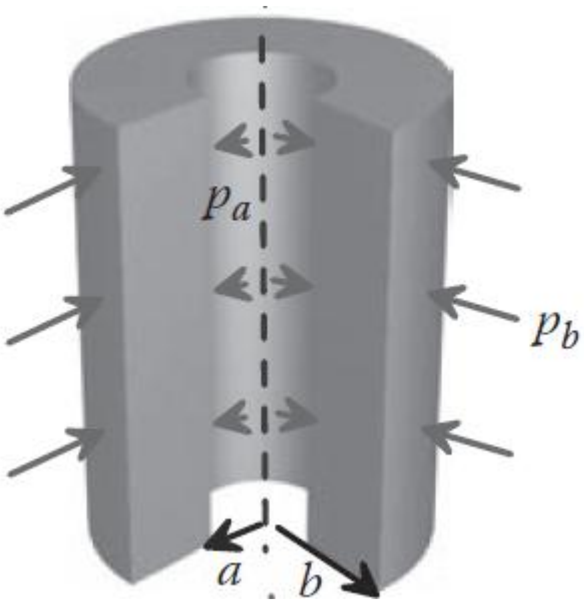
$$\Rightarrow \frac{d^2u_r}{dr^2}-\frac{u_r}{r^2}+\frac{1}{r}\frac{du_r}{dr}=\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(ru_r)\right\}=\alpha(1+\nu)\frac{d\Delta T}{dr}-\frac{(1-\nu^2)}{E}(F_r+\rho\omega^2 r).$$

- Given the temperature and/or body force distributions, the radial displacement can be solved by integration.
- Two constants of integrations must be determined from BCs.
- For the generalized plane strain solution, ε_{zz} can be determined from:

$$F_z=\int_a^b 2\pi r\sigma_z dr,\quad \sigma_z=\frac{E}{(1+\nu)(1-2\nu)}\{\nu\varepsilon_r+\nu\varepsilon_\theta+(1-\nu)\varepsilon_z\}-\frac{E\alpha\Delta T}{(1-2\nu)}.$$

Pressurized Cylindrical Shell

- No body forces act on the cylinder.
- The cylinder has zero angular velocity.
- The cylinder has uniform temperature.



- **Plane strain solution:**

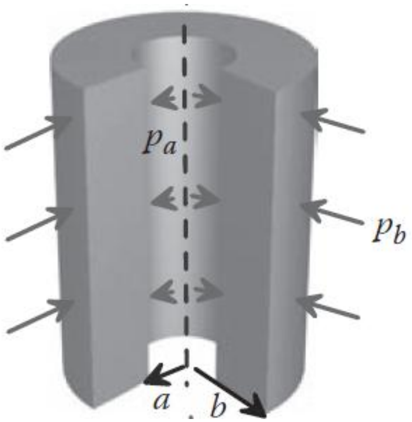
$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(ru_r\right)\right\}=0 \quad \Rightarrow u_r = Ar+\frac{B}{r}$$

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\frac{du_r}{dr}+\nu\frac{u_r}{r}\right\}=\frac{E}{(1+\nu)(1-2\nu)}\left\{A-(1-2\nu)\frac{B}{r^2}\right\}$$

$$\left.\begin{matrix}\sigma_r[a]=-p_a\\ \sigma_r[b]=-p_b\end{matrix}\right\}\Rightarrow\left\{A=\frac{(1+\nu)(1-2\nu)}{E}\frac{\left(p_a a^2-p_b b^2\right)}{b^2-a^2},\quad B=\frac{(1+\nu)}{E}\frac{a^2b^2\left(p_a-p_b\right)}{b^2-a^2}\right.$$.

Pressurized Cylindrical Shell

- **Plane strain solution:**

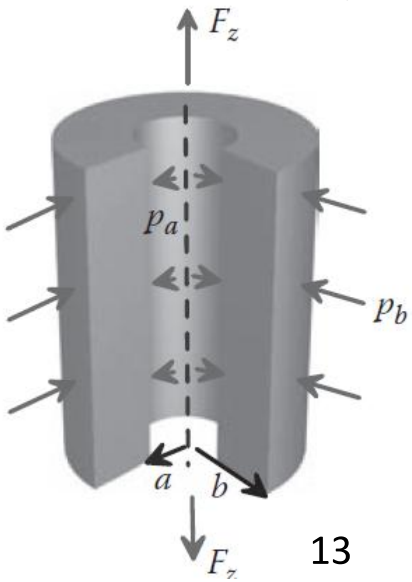


$$\sigma_r = \frac{\left(p_a a^2 - p_b b^2\right)}{b^2 - a^2} - \frac{a^2 b^2 \left(p_a - p_b\right)}{r^2 \left(b^2 - a^2\right)}, \sigma_\theta = \frac{\left(p_a a^2 - p_b b^2\right)}{b^2 - a^2} + \frac{a^2 b^2 \left(p_a - p_b\right)}{r^2 \left(b^2 - a^2\right)}$$

$$\varepsilon_z = 0 \quad \Rightarrow \sigma_z = \nu\left(\sigma_r + \sigma_\theta\right) = \frac{2\nu\left(p_a a^2 - p_b b^2\right)}{b^2 - a^2};$$

$$u_r = \frac{(1+\nu)}{E\left(b^2 - a^2\right)}\left\{(1-2\nu)\left(p_a a^2 - p_b b^2\right) r + \frac{a^2 b^2 \left(p_a - p_b\right)}{r}\right\}.$$

- **If the cylinder is allowed to stretch parallel to its axis:**



$$F_z = \int_a^b 2\pi r \sigma_z dr = \pi\left(p_a a^2 - p_b b^2\right)$$

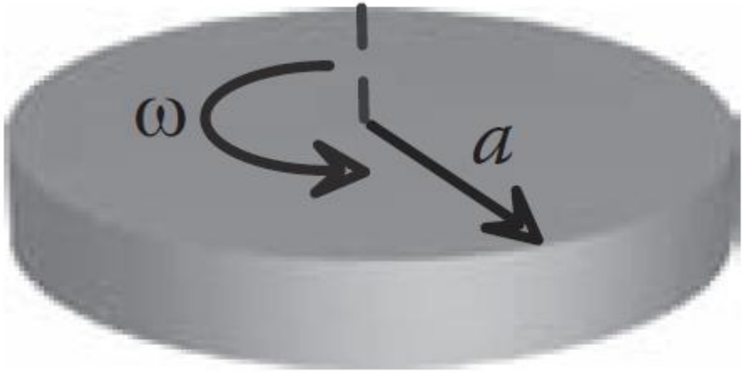
$$\Rightarrow \varepsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}\left(\sigma_r + \sigma_\theta\right) = \frac{F_z}{E\pi\left(b^2 - a^2\right)} - \frac{2\nu\left(p_a a^2 - p_b b^2\right)}{E\left(b^2 - a^2\right)};$$

$$\Rightarrow \mathbf{u} = \frac{(1+\nu)}{E\left(b^2 - a^2\right)}\left\{(1-2\nu)\left(p_a a^2 - p_b b^2\right) r + \frac{a^2 b^2 \left(p_a - p_b\right)}{r}\right\}\mathbf{e}_r$$

$$-\nu\varepsilon_z r\mathbf{e}_r + \varepsilon_z z\mathbf{e}_z.$$

Spinning Disk

- No body forces act on the disk.
- The disk has uniform temperature.
- The disk is sufficiently thin to ensure a state of plane stress in the disk.

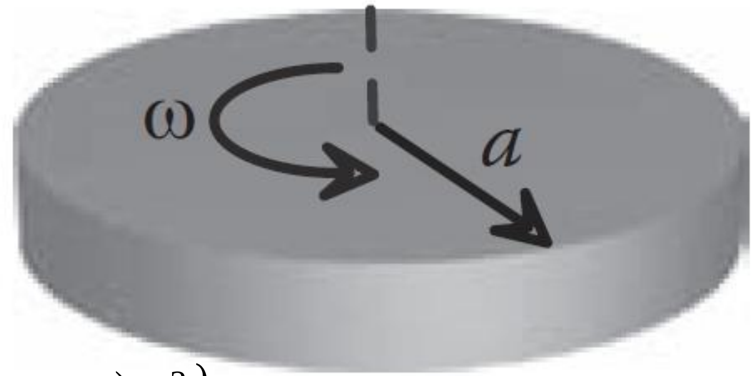


$$\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(ru_r\right)\right\} = -\frac{\left(1-\nu^2\right)}{E}\rho\omega^2 r \quad \Rightarrow u_r = Ar + \frac{B}{r} - \frac{\left(1-\nu^2\right)}{8E}\rho\omega^2 r^3$$

$$\sigma_r = \frac{E}{1-\nu^2}\left\{\frac{\mathrm{d}u_r}{\mathrm{d}r} + \nu\frac{u_r}{r}\right\} = \frac{E}{1-\nu^2}\left\{\left(1+\nu\right)A - \left(1-\nu\right)\frac{B}{r^2} - \frac{\left(1-\nu^2\right)\left(3+\nu\right)}{8E}\rho\omega^2 r^2\right\}$$

$$\sigma_r\left[0\right] = \text{finite} \quad \Rightarrow B = 0; \qquad \sigma_r\left[a\right] = 0 \quad \Rightarrow A = \frac{\left(1-\nu\right)\left(3+\nu\right)}{8E}\rho\omega^2 a^2.$$

Spinning Disk



$$u_r = \frac{(1-\nu)\rho\omega^2}{8E}\left\{(3+\nu)a^2 r - (1+\nu)r^3\right\};$$

$$\sigma_r = \frac{(3+\nu)}{8}\rho\omega^2\left(a^2 - r^2\right),$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left\{\nu\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r}\right\} = \frac{\rho\omega^2}{8}\left\{(3+\nu)a^2 - (3\nu+1)r^2\right\};$$

$$\varepsilon_z = \frac{\cancel{\sigma_z}}{E} - \frac{\nu}{E}\left(\sigma_r + \sigma_\theta\right) = -\frac{\nu\rho\omega^2}{4E}\left\{(3+\nu)a^2 - 2(1+\nu)r^2\right\};$$

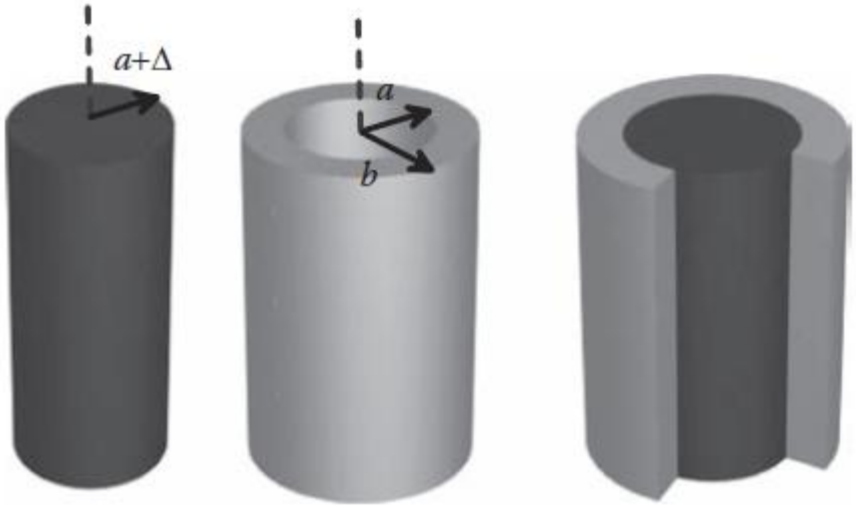
$$u_z = z\varepsilon_z = -\frac{\nu\rho\omega^2 z}{4E}\left\{(3+\nu)a^2 - 2(1+\nu)r^2\right\}.$$

- The maximum stress occurs at the center of the disk, even though the centrifugal force is largest at the outer boundary.

$$\sigma_{\max} = \sigma_r[0] = \sigma_\theta[0] = \frac{(3+\nu)}{8}\rho\omega^2 a^2.$$

Interference Fit between Two Cylinders

- No body forces act on the solids.
- The cylinders have uniform temperature.
- Generalized plane strain



- The axial force acting on both the shaft and the tube vanish separately, if they slide freely relative to one another.
- After the shaft is inserted into the tube, a pressure p acts to compress the shaft, and the same pressure pushes outward to expand the cylinder.

$$u_r[a] - \bar{u}_r[a] = \Delta.$$

Interference Fit between Two Cylinders

- Displacement BCs

$$\bar{\varepsilon}_z = \frac{2\nu p}{E}; \quad \bar{u}_r = -\frac{(1+\nu)(1-2\nu)}{E} pr - \frac{2\nu^2 pr}{E};$$

$$\varepsilon_z = -\frac{2\nu pa^2}{E(b^2-a^2)}; \quad u_r = \frac{(1+\nu)}{E(b^2-a^2)}\left\{(1-2\nu)pa^2 r + \frac{a^2b^2p}{r}\right\} + \frac{2\nu^2 pa^2 r}{E(b^2-a^2)}.$$

$$u_r[a] - \bar{u}_r[a] = \Delta \qquad \Rightarrow \boxed{p = \frac{E\Delta(b^2-a^2)}{2ab^2}}$$

- In the shaft

$$\bar{\sigma}_r = \bar{\sigma}_\theta = -p, \bar{\sigma}_z = 0. \qquad \Rightarrow \boxed{\bar{\sigma}_r = \bar{\sigma}_\theta = -\frac{E\Delta(b^2-a^2)}{2ab^2}, \bar{\sigma}_z = 0}$$

$$\Rightarrow \bar{\mathbf{u}} = -\frac{(1+\nu)(1-2\nu)}{E} pr\mathbf{e}_r - \frac{2\nu^2 pr}{E}\mathbf{e}_r + \frac{2\nu pz}{E}\mathbf{e}_z$$

$$\Rightarrow \boxed{\bar{\mathbf{u}} = -\frac{(1-\nu)\Delta(b^2-a^2)}{2ab^2} r\mathbf{e}_r + \frac{\nu\Delta(b^2-a^2)z}{ab^2}\mathbf{e}_z}$$

Interference Fit between Two Cylinders

- In the cylinder

$$\sigma_r = \frac{pa^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right), \sigma_\theta = \frac{pa^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right), \sigma_z = 0.$$

$$\Rightarrow \boxed{\sigma_r = \frac{Ea\Delta}{2b^2}\left(1 - \frac{b^2}{r^2}\right), \sigma_\theta = \frac{Ea\Delta}{2b^2}\left(1 + \frac{b^2}{r^2}\right), \sigma_z = 0}$$

$$\Rightarrow \mathbf{u} = \frac{(1+\nu)\, pa^2 r}{E\left(b^2 - a^2\right)}\left((1-2\nu) + \frac{b^2}{r^2}\right)\mathbf{e}_r + \frac{2\nu^2 pa^2 r}{E\left(b^2 - a^2\right)}\mathbf{e}_r - \frac{2\nu\, pa^2 z}{E\left(b^2 - a^2\right)}\mathbf{e}_z$$

$$\Rightarrow \boxed{\mathbf{u} = \frac{(1+\nu)\Delta ar}{2b^2}\left(1 - 2\nu + \frac{b^2}{r^2}\right)\mathbf{e}_r + \frac{\nu^2 \Delta ar}{b^2}\mathbf{e}_r - \frac{\nu\Delta az}{b^2}\mathbf{e}_z}$$

Small Strain and Rotation in Spherical Coordinates

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u}\bar{\nabla} + \nabla\mathbf{u}\right); \quad \boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{u}\bar{\nabla} - \nabla\mathbf{u}\right); \ \mathbf{u} = u_R\mathbf{e}_R + u_\varphi\mathbf{e}_\varphi + u_\theta\mathbf{e}_\theta;$$

$$\mathbf{u}\bar{\nabla}_s = \begin{bmatrix} \dfrac{\partial u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \left(\dfrac{1}{R}\dfrac{\partial u_R}{\partial\varphi} - \dfrac{u_\varphi}{R}\right)\mathbf{e}_R\mathbf{e}_\varphi + \left(\dfrac{1}{R\sin\varphi}\dfrac{\partial u_R}{\partial\theta} - \dfrac{u_\theta}{R}\right)\mathbf{e}_R\mathbf{e}_\theta \\ +\dfrac{\partial u_\varphi}{\partial R}\mathbf{e}_\varphi\mathbf{e}_R + \left(\dfrac{u_R}{R} + \dfrac{1}{R}\dfrac{\partial u_\varphi}{\partial\varphi}\right)\mathbf{e}_\varphi\mathbf{e}_\varphi + \left(-\dfrac{\cot\varphi u_\theta}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\varphi}{\partial\theta}\right)\mathbf{e}_\varphi\mathbf{e}_\theta \\ +\dfrac{\partial u_\theta}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \dfrac{1}{R}\dfrac{\partial u_\theta}{\partial\varphi}\mathbf{e}_\theta\mathbf{e}_\varphi + \left(\dfrac{u_R}{R} + \dfrac{1}{R\sin\varphi}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{\cot\varphi u_\varphi}{R}\right)\mathbf{e}_\theta\mathbf{e}_\theta \end{bmatrix}$$

$$\omega_R = \omega_\theta = \omega_\varphi = 0, \omega_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R} - \frac{\partial u_\theta}{\partial R}\right),$$

$$\omega_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial\varphi} + \frac{\cot\varphi u_\theta}{R} - \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial\theta}\right), \omega_{\varphi R} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial R} - \frac{1}{R}\frac{\partial u_R}{\partial\varphi} + \frac{u_\varphi}{R}\right);$$

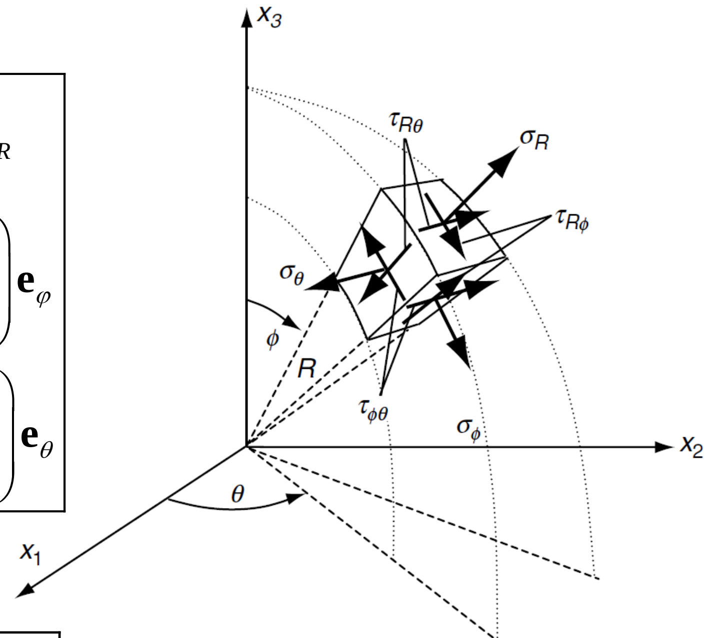
$$\varepsilon_R = \frac{\partial u_R}{\partial R}, \varepsilon_\varphi = \frac{u_R}{R} + \frac{1}{R}\frac{\partial u_\varphi}{\partial\varphi}, \varepsilon_\theta = \frac{u_R}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\theta}{\partial\theta} + \frac{\cot\varphi u_\varphi}{R}, \varepsilon_{R\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial R} + \frac{1}{R}\frac{\partial u_R}{\partial\varphi} - \frac{u_\varphi}{R}\right),$$

$$\varepsilon_{R\theta} = \frac{1}{2}\left(\frac{1}{R\sin\varphi}\frac{\partial u_R}{\partial\theta} - \frac{u_\theta}{R} + \frac{\partial u_\theta}{\partial R}\right), \varepsilon_{\varphi\theta} = \frac{1}{2}\left(\frac{1}{R}\frac{\partial u_\theta}{\partial\varphi} - \frac{\cot\varphi u_\theta}{R} + \frac{1}{R\sin\varphi}\frac{\partial u_\varphi}{\partial\theta}\right).$$

Equilibrium Equations in Spherical Coordinates

$\nabla\cdot\boldsymbol{\sigma}$ = contraction on the first and third index of $\boldsymbol{\sigma}\bar{\nabla}$

$$\Rightarrow\quad \boxed{\begin{aligned}\nabla\cdot\boldsymbol{\sigma} = &\left(\frac{\partial\sigma_R}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\varphi R}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta R}}{\partial\theta}+\frac{2\sigma_R-\sigma_\theta-\sigma_\varphi+\cot\varphi\tau_{\varphi R}}{R}\right)\mathbf{e}_R\\ &+\left(\frac{\partial\tau_{R\varphi}}{\partial R}+\frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{2\tau_{R\varphi}+\tau_{\varphi R}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R}\right)\mathbf{e}_\varphi\\ &+\left(\frac{\partial\tau_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\varphi\theta}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{R\theta}+\tau_{\theta R}+\cot\varphi\left(\tau_{\theta\varphi}+\tau_{\varphi\theta}\right)}{R}\right)\mathbf{e}_\theta\end{aligned}}$$



$\nabla\cdot\boldsymbol{\sigma}+\mathbf{F}=\mathbf{0}$

$$\boxed{\begin{aligned}&\frac{\partial\sigma_R}{\partial R}+\frac{1}{R}\frac{\partial\tau_{R\varphi}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{R\theta}}{\partial\theta}+\frac{2\sigma_R-\sigma_\theta-\sigma_\varphi+\cot\varphi\tau_{R\varphi}}{R}+F_R=0,\\ &\frac{\partial\tau_{R\varphi}}{\partial R}+\frac{1}{R}\frac{\partial\sigma_\varphi}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\tau_{\theta\varphi}}{\partial\theta}+\frac{3\tau_{R\varphi}-\cot\varphi\left(\sigma_\theta-\sigma_\varphi\right)}{R}+F_\varphi=0,\\ &\frac{\partial\tau_{R\theta}}{\partial R}+\frac{1}{R}\frac{\partial\tau_{\theta\varphi}}{\partial\varphi}+\frac{1}{R\sin\varphi}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{3\tau_{R\theta}+2\tau_{\theta\varphi}\cot\varphi}{R}+F_\theta=0.\end{aligned}}$$

$$\boldsymbol{\sigma}=\begin{bmatrix}\sigma_R & \tau_{R\varphi} & \tau_{R\theta}\\ \tau_{R\varphi} & \sigma_\varphi & \tau_{\theta\varphi}\\ \tau_{R\theta} & \tau_{\theta\varphi} & \sigma_\theta\end{bmatrix}$$

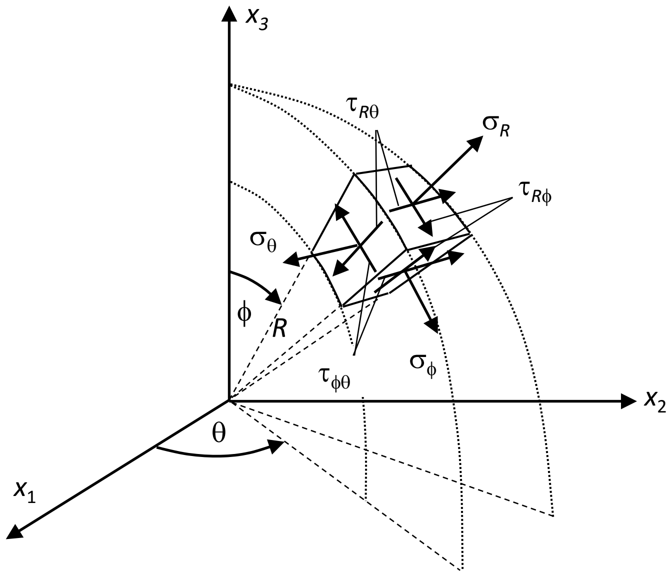
$$\mathbf{T}^{\mathbf{R}}=\sigma_R\mathbf{e}_R+\tau_{R\varphi}\mathbf{e}_\varphi+\tau_{R\theta}\mathbf{e}_\theta$$

$$\mathbf{T}^{\boldsymbol{\varphi}}=\tau_{R\varphi}\mathbf{e}_R+\sigma_\varphi\mathbf{e}_\varphi+\tau_{\theta\varphi}\mathbf{e}_\theta$$

$$\mathbf{T}^{\boldsymbol{\theta}}=\tau_{R\theta}\mathbf{e}_R+\tau_{\theta\varphi}\mathbf{e}_\varphi+\sigma_\theta\mathbf{e}_\theta$$

Hooke's Law in Spherical Coordinates

- Recall that, **the elastic stiffness tensor *C* is a fourth order isotropic tensor**.
- Its components remain unchanged under any orthogonal coordinate systems.
- The isotropic Hooke's law stays the same.



$$\varepsilon_{ij}^{\text{Total}} = \varepsilon_{ij}^{M} + \varepsilon_{ij}^{T} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij},$$

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right\} - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij}.$$

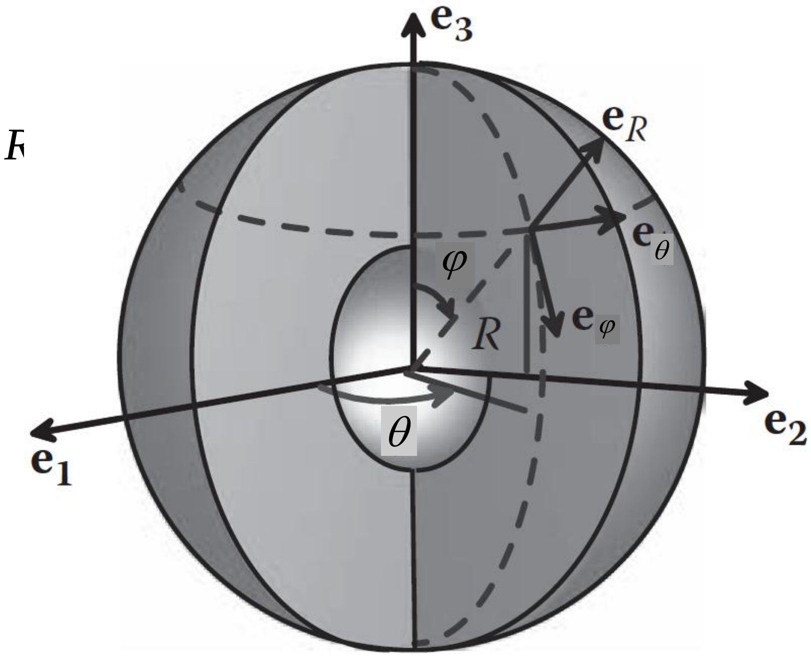
$$\sigma_R = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + \varepsilon_R\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_\varphi = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + \varepsilon_\varphi\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\sigma_\theta = \frac{E}{(1+\nu)}\left\{\frac{\nu}{1-2\nu}\left(\varepsilon_R + \varepsilon_\varphi + \varepsilon_\theta\right) + \varepsilon_\theta\right\} - \frac{E\alpha\Delta T}{(1-2\nu)},$$

$$\tau_{R\varphi} = \frac{E}{(1+\nu)}\varepsilon_{R\varphi}, \tau_{R\theta} = \frac{E}{(1+\nu)}\varepsilon_{R\theta}, \tau_{\theta\varphi} = \frac{E}{(1+\nu)}\varepsilon_{\theta\varphi}$$

Spherical Symmetry



- Displacements and stresses

$$\mathbf{u} = u_R[R]\mathbf{e}_R, \quad \boldsymbol{\sigma} = \sigma_R[R]\mathbf{e}_R\mathbf{e}_R + \sigma_\varphi[R]\mathbf{e}_\varphi\mathbf{e}_\varphi + \sigma_\theta[R$$

- Strain-displacement relation:

$$\varepsilon_R = \frac{\mathrm{d}u_R}{\mathrm{d}R}, \quad \varepsilon_\varphi = \varepsilon_\theta = \frac{u_R}{R}$$

- Equilibrium equations

$$\frac{d\sigma_R}{dR} + \frac{2}{R}\left(\sigma_R - \sigma_\varphi\right) + F_R = 0.$$

- Hooke's law

$$\begin{cases} \sigma_R = \dfrac{E}{(1+\nu)}\left\{\dfrac{\nu}{1-2\nu}\left(\varepsilon_R + 2\varepsilon_\varphi\right) + \varepsilon_R\right\} - \dfrac{E\alpha\Delta T}{(1-2\nu)} \\ \sigma_\varphi = \dfrac{E}{(1+\nu)}\left\{\dfrac{\nu}{1-2\nu}\left(\varepsilon_R + 2\varepsilon_\varphi\right) + \varepsilon_\varphi\right\} - \dfrac{E\alpha\Delta T}{(1-2\nu)} \end{cases}$$

$$\Rightarrow \begin{Bmatrix} \sigma_R \\ \sigma_\varphi \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_R \\ \varepsilon_\varphi \end{Bmatrix} - \frac{E\alpha\Delta T}{(1-2\nu)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- Boundary conditions

$$u_R[a] = u_a, u_R[b] = u_b$$

$$\sigma_R[a] = \sigma_a, \sigma_R[b] = \sigma_b$$

Spherical Symmetry

- Stresses in terms of displacements

$$\begin{Bmatrix}\sigma_R\\ \sigma_\varphi\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & 2\nu\\ \nu & 1\end{bmatrix}\begin{Bmatrix}\dfrac{\mathrm{d}u_R}{\mathrm{d}R}\\ \dfrac{u_R}{R}\end{Bmatrix}-\frac{E\alpha\Delta T}{(1-2\nu)}\begin{Bmatrix}1\\1\end{Bmatrix}.$$

- Equilibrium equations in terms of displacements

$$\sigma_R-\sigma_\varphi=\frac{E}{(1+\nu)}\left\{\frac{\mathrm{d}u_R}{\mathrm{d}R}-\frac{u_R}{R}\right\},\ \frac{d\sigma_R}{dR}=\frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\frac{\mathrm{d}^2u_R}{\mathrm{d}R^2}+2\nu\left(\frac{1}{R}\frac{\mathrm{d}u_R}{\mathrm{d}R}-\frac{u_R}{R^2}\right)\right\}-\frac{E\alpha}{(1-2\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}R}$$

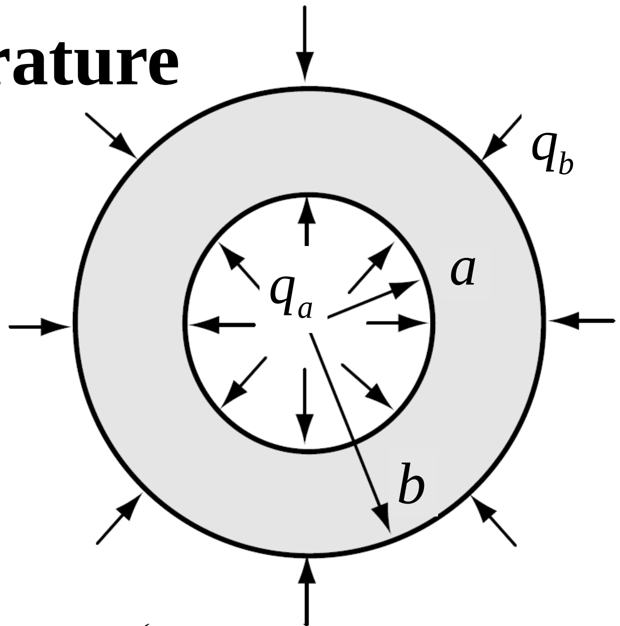
$$\frac{d\sigma_R}{dR}+\frac{2}{R}(\sigma_R-\sigma_\varphi)+F_R=0\Rightarrow\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left\{\frac{\mathrm{d}^2u_R}{\mathrm{d}R^2}+\frac{2}{R}\frac{\mathrm{d}u_R}{\mathrm{d}R}-\frac{2u_R}{R^2}\right\}-\frac{E\alpha}{(1-2\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}R}+F_R=0$$

$$\Rightarrow\boxed{\frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2u_R\right)\right\}=\frac{(1+\nu)\alpha}{(1-\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}R}-\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}F_R}$$

- Given the temperature and/or body force distributions, the radial displacement can be solved by integration.
- Two constants of integrations must be determined from BCs.

Pressurized Spherical Shell

- **No body forces and uniform temperature**



$$\frac{d}{dR}\left(\frac{1}{R^2}\frac{d}{dR}\left(R^2u_R\right)\right)=0$$

$$\Rightarrow \frac{1}{R^2}\frac{d}{dR}\left(R^2u_R\right)=3A \quad \Rightarrow \quad \frac{d}{dR}\left(R^2u_R\right)=3AR^2$$

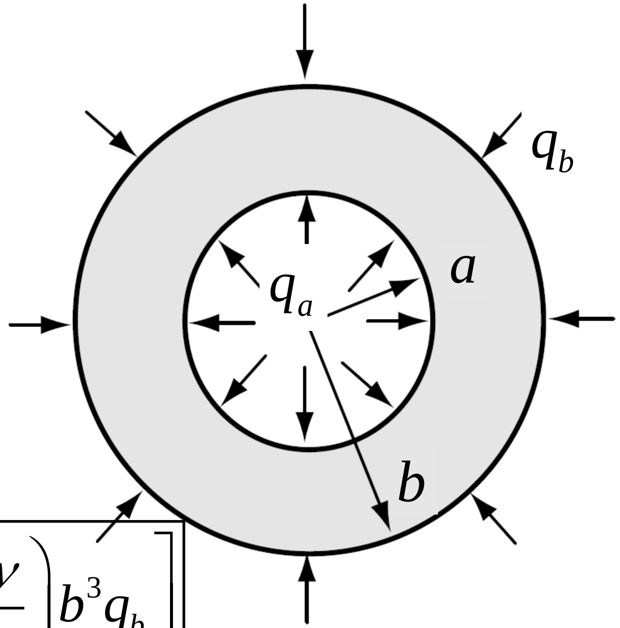
$$\Rightarrow R^2u_R = AR^3+B \quad \Rightarrow \quad u_R = AR+\frac{B}{R^2}$$

$$\Rightarrow \begin{cases} \sigma_R = \lambda\left(\frac{\partial u_R}{\partial R}+\frac{2u_R}{R}\right)+2G\frac{\partial u_R}{\partial R} = \lambda\left(A-\frac{2B}{R^3}+2A+\frac{2B}{R^3}\right)+2G\left(A-\frac{2B}{R^3}\right) \\ \sigma_\varphi = \sigma_\theta = \lambda\left(\frac{\partial u_R}{\partial R}+\frac{2u_R}{R}\right)+2G\frac{u_R}{R} = \lambda\left(A-\frac{2B}{R^3}+2A+2\frac{B}{R^3}\right)+2G\left(A+\frac{B}{R^3}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_R = \left(3\lambda+2G\right)A-\frac{4G}{R^3}B = \frac{E}{\left(1-2\nu\right)}A-\frac{2E}{\left(1+\nu\right)}\frac{1}{R^3}B \\ \sigma_\varphi = \sigma_\theta = \left(3\lambda+2G\right)A+\frac{2G}{R^3}B = \frac{E}{\left(1-2\nu\right)}A+\frac{E}{\left(1+\nu\right)}\frac{1}{R^3}B \end{cases}$$

Pressurized Spherical Shell

- The traction BCs at $R = a$ and $R = b$



$$\begin{cases} -q_a = (\sigma_R)_{R=a} = \dfrac{E}{1-2\nu}A - \dfrac{2E}{1+\nu}\dfrac{B}{a^3} \\ -q_b = (\sigma_R)_{R=b} = \dfrac{E}{1-2\nu}A + \dfrac{E}{1+\nu}\dfrac{B}{b^3} \end{cases}$$

$$\Rightarrow A = \frac{(1-2\nu)\left(a^3 q_a - b^3 q_b\right)}{E\left(b^3 - a^3\right)}, \quad B = \frac{(1+\nu)(q_a - q_b)a^3 b^3}{2E\left(b^3 - a^3\right)}$$

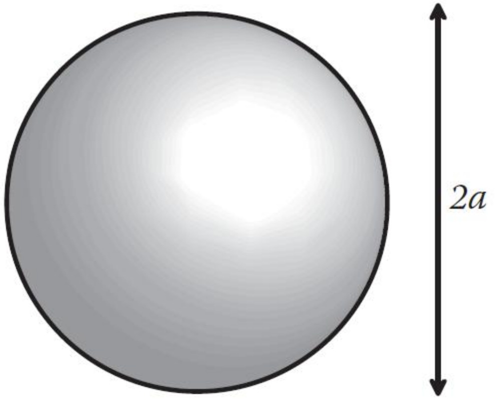
$$\Rightarrow \boxed{u_R = \frac{(1+\nu)}{E\left(b^3 - a^3\right)} R \left[\left(\frac{b^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right)a^3 q_a - \left(\frac{a^3}{2R^3} + \frac{1-2\nu}{1+\nu}\right)b^3 q_b\right]}$$

$$\sigma_R = -\frac{1}{b^3 - a^3}\left(\frac{b^3}{R^3} - 1\right)a^3 q_a - \frac{1}{b^3 - a^3}\left(1 - \frac{a^3}{R^3}\right)b^3 q_b,$$

$$\sigma_\varphi = \sigma_\theta = \frac{1}{b^3 - a^3}\left(\frac{b^3}{2R^3} + 1\right)a^3 q_a - \frac{1}{b^3 - a^3}\left(1 + \frac{a^3}{2R^3}\right)b^3 q_b$$

- **Here, stress is independent of Poisson's ratio. However, generally in 3-D problems with specified tractions, stress depends on Poisson's ratio.**

Gravitating Planet



- Subject to its own gravitation attraction
- Uniform temperature
- Traction free at the surface of the sphere

$$\mathbf{F} = -\rho g R / a\, \mathbf{e}_R, \quad \frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2 u_R\right)\right\} = \frac{(1+\nu)\alpha}{(1-\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}R} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}F_R$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2 u_R\right)\right\} = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{\rho g R}{a} \quad \Rightarrow u_R = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{\rho g R^3}{10a} + AR + \frac{B}{R^2}$$

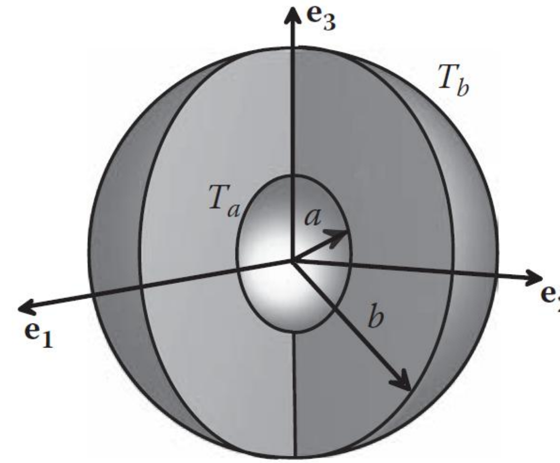
$$\Rightarrow \sigma_R = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\frac{\mathrm{d}u_R}{\mathrm{d}R} + 2\nu\frac{u_R}{R}\right\} = \frac{\rho g(3-\nu)R^2}{10a(1-\nu)} + \frac{E}{(1+\nu)(1-2\nu)}\left\{(1+\nu)A - 2(1-2\nu)\frac{B}{R^3}\right\}$$

- Expect finite displacement and stress at the center (R=0).

$$\sigma_R[a] = 0 \quad \Rightarrow A = -\frac{\rho g(1-2\nu)(3-\nu)a}{10E(1-\nu)} \quad \Rightarrow \boxed{u_R = \frac{(1-2\nu)\rho g R}{10aE(1-\nu)}\left\{(1+\nu)R^2 - (3-\nu)a^2\right\}}$$

$$\boxed{\sigma_R = \frac{\rho g(3-\nu)}{10a(1-\nu)}\left(R^2 - a^2\right), \quad \sigma_\varphi = \sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\frac{\mathrm{d}u_R}{\mathrm{d}R} + \frac{u_R}{R}\right\} = \frac{\rho g}{10a(1-\nu)}\left\{(1+3\nu)R^2 - (3-\nu)a^2\right\}}$$

Steady-state Heat Flow



- No body forces
- Free of tractions at both surfaces

$$T=\frac{T_b b-T_a a}{b-a}+\frac{(T_a-T_b)ab}{(b-a)R},$$

$$\frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2u_R\right)\right\}=\frac{(1+\nu)\alpha}{(1-\nu)}\frac{\mathrm{d}\Delta T}{\mathrm{d}R}-\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}F_R$$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2u_R\right)\right\}=-\frac{(1+\nu)\alpha}{(1-\nu)}\frac{(T_a-T_b)ab}{(b-a)R^2}\quad\Rightarrow u_R=\frac{(1+\nu)\alpha}{2(1-\nu)}\frac{(T_a-T_b)ab}{(b-a)}+AR+\frac{B}{R^2}$$

$$\Rightarrow\sigma_R=\frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\frac{\mathrm{d}u_R}{\mathrm{d}R}+2\nu\frac{u_R}{R}\right\}-\frac{E\alpha\Delta T}{(1-2\nu)}$$

$$\Rightarrow\sigma_R=\frac{E\nu\alpha}{(1-2\nu)(1-\nu)}\frac{(T_a-T_b)ab}{(b-a)R}+\frac{E}{(1+\nu)(1-2\nu)}\left\{(1+\nu)A-2(1-2\nu)\frac{B}{R^3}\right\}-\frac{E\alpha}{(1-2\nu)}\left\{\frac{T_b b-T_a a}{b-a}+\frac{(T_a-T_b)ab}{(b-a)R}\right\}$$

$$\sigma_R[a]=\sigma_R[b]=0\quad\Rightarrow$$

$$A=\frac{(1-\nu)\left(T_b b^3-T_a a^3\right)+(T_a-T_b)\nu ab(a+b)}{(1-\nu)\left(a^3-b^3\right)}\qquad B=\frac{\alpha(T_a-T_b)(1+\nu)}{2(1-\nu)}\frac{a^3b^3}{\left(b^3-a^3\right)}$$