# Equivalent Systems of Forces

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# Introduction

- Treatment of a body as a single particle is not always possible. In general, the size of the body and the specific points of application of the forces must be considered.

- Most bodies in elementary mechanics are assumed to be rigid, i.e., the actual deformations are small and do not affect the conditions of equilibrium or motion of the body.

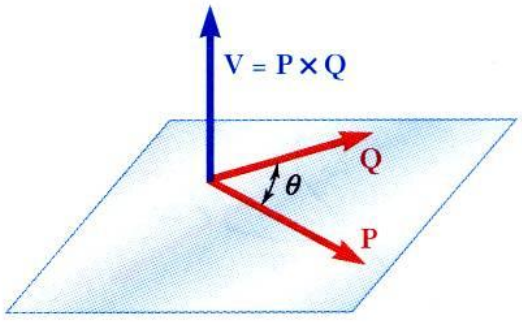
- Current chapter describes the effect of forces exerted on a rigid body and how to replace a given system of forces with a simpler equivalent system.
  - moment of a force about a point
  - moment of a force about an axis
  - moment due to a couple

- Any system of forces acting on a rigid body can be replaced by an equivalent system consisting of one force acting at a given point and one couple.

# Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is $V = PQ\sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.




(a)

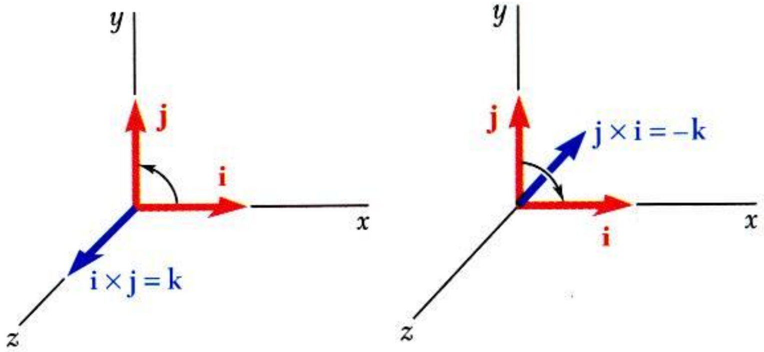

(b)

- Vector products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

# Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors,

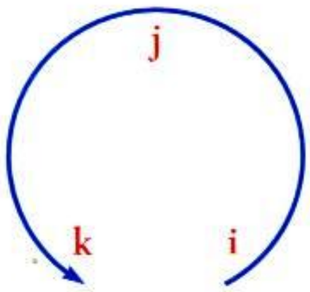
$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$




- Vector products in terms of rectangular coordinates

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$
$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$



# Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on its point of application.

- The *moment* of $\boldsymbol{F}$ about $O$ is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

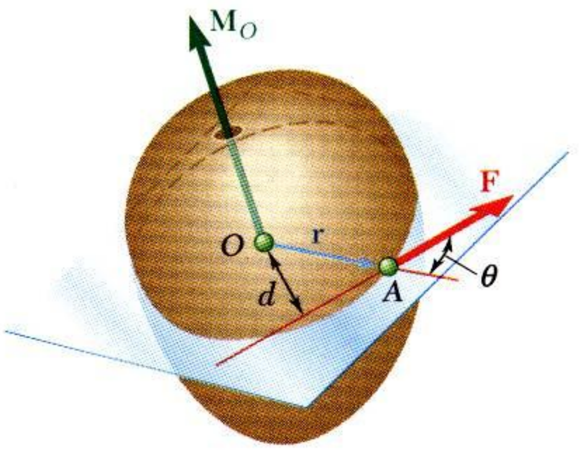
- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing $O$ and the force $\boldsymbol{F}$.

- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF\sin\theta = Fd$$

  The sense of the moment may be determined by the right-hand rule.

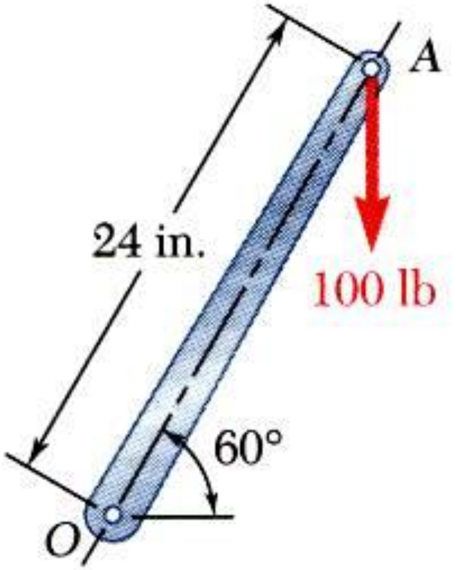
- Any force $\boldsymbol{F'}$ that has the same magnitude and direction as $\boldsymbol{F}$, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



(a)

(b)

# Sample Problem




A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.

Determine:

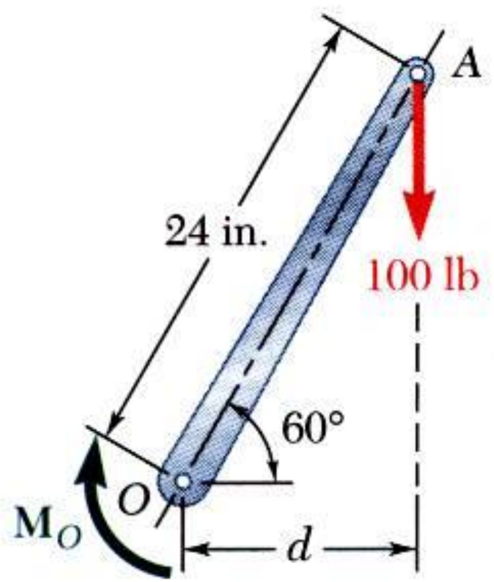
a) moment about *O*,

b) horizontal force at *A* which creates the same moment,

c) smallest force at A which produces the same moment,

d) location for a 240-lb vertical force to produce the same moment,

e) whether any of the forces from b, c, and d is equivalent to the original force.



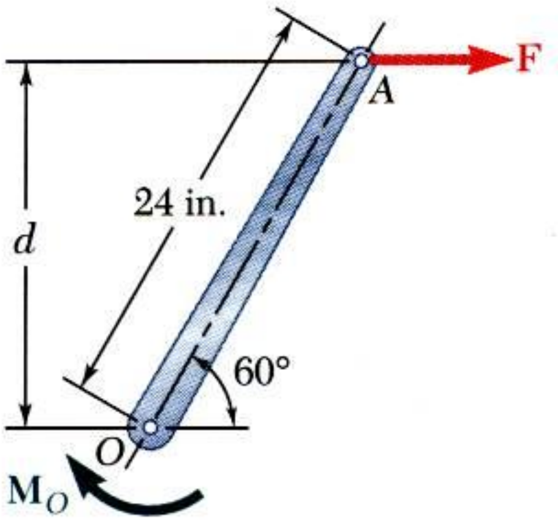

a) Moment about $O$ is equal to the product of the force and the perpendicular distance between the line of action of the force and $O$. Since the force tends to rotate the lever clockwise, the moment vector is into the plane of the paper.

$$M_O = Fd$$

$$d = (24\,\text{in.})\cos 60^\circ = 12\,\text{in.}$$

$$M_O = (100\,\text{lb})(12\,\text{in.})$$

$$\boxed{M_O = 1200\,\text{lb}\cdot\text{in}}$$




c) Horizontal force at *A* that produces the same moment,

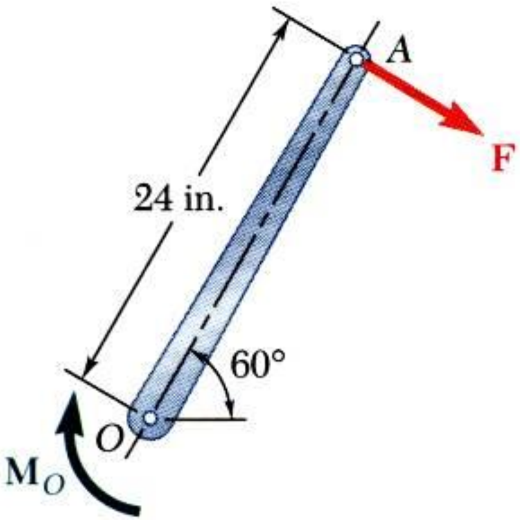
$$d = (24\text{ in.})\sin 60° = 20.8\text{ in.}$$

$$M_O = Fd$$

$$1200\text{ lb}\cdot\text{in.} = F(20.8\text{ in.})$$

$$F = \frac{1200\text{ lb}\cdot\text{in.}}{20.8\text{ in.}}$$

$$\boxed{F = 57.7\text{ lb}}$$



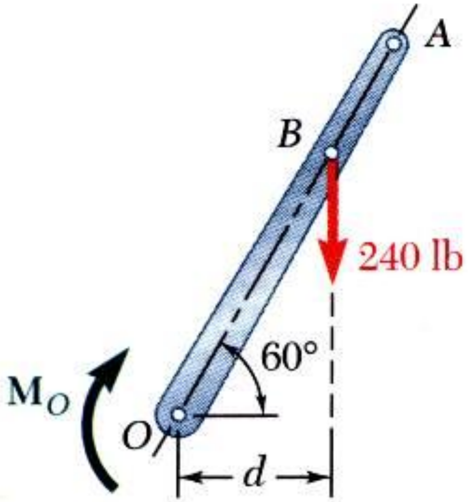

c) The smallest force *A* to produce the same moment occurs when the perpendicular distance is a maximum or when *F* is perpendicular to *OA*.

$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = F(24\,\text{in.})$$

$$F = \frac{1200\,\text{lb}\cdot\text{in.}}{24\,\text{in.}}$$

$$\boxed{F = 50\,\text{lb}}$$




d) To determine the point of application of a 240 lb force to produce the same moment,

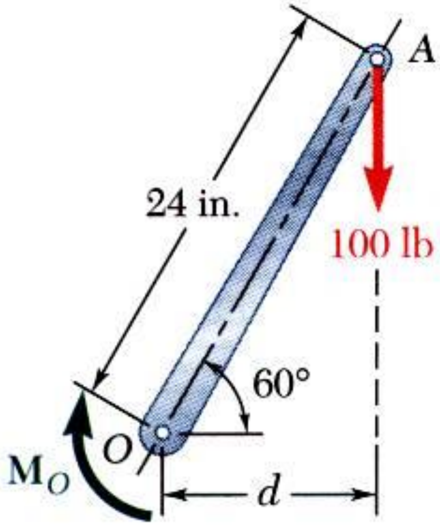
$$M_O = Fd$$

$$1200\,\text{lb}\cdot\text{in.} = (240\,\text{lb})d$$

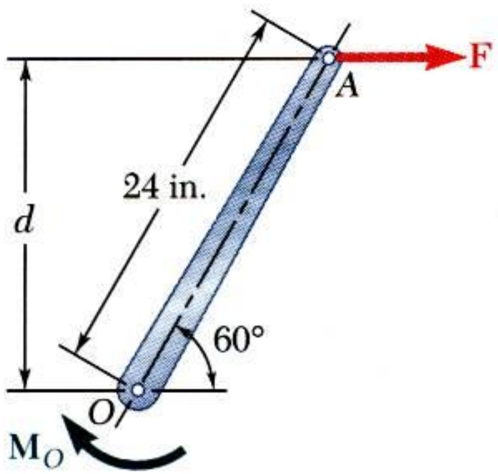
$$d = \frac{1200\,\text{lb}\cdot\text{in.}}{240\,\text{lb}} = 5\,\text{in.}$$

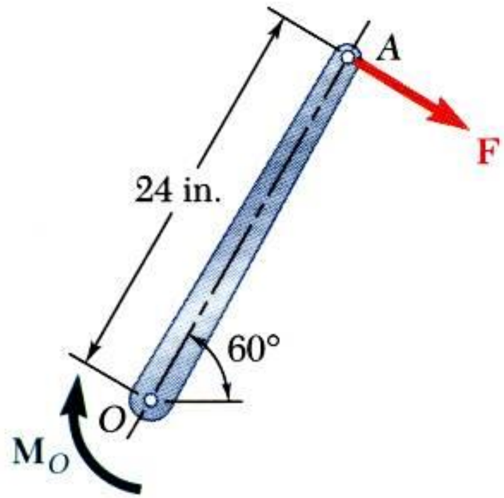
$$OB\cos 60° = 5\,\text{in.}$$

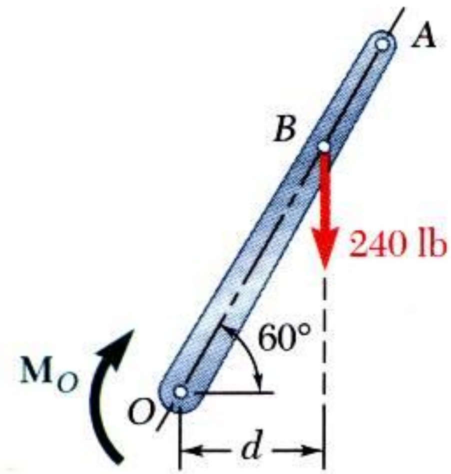
$$\boxed{OB = 10\,\text{in.}}$$




e) Although each of the forces in parts b), c), and d) produces the same moment as the 100 lb force, none are of the same magnitude and sense, or on the same line of action. None of the forces is equivalent to the 100 lb force.



# Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment $M_{OL}$ about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

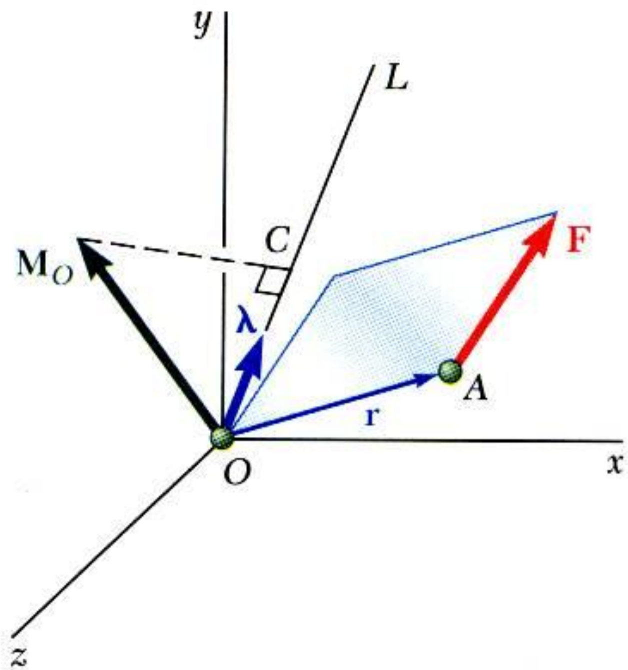
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

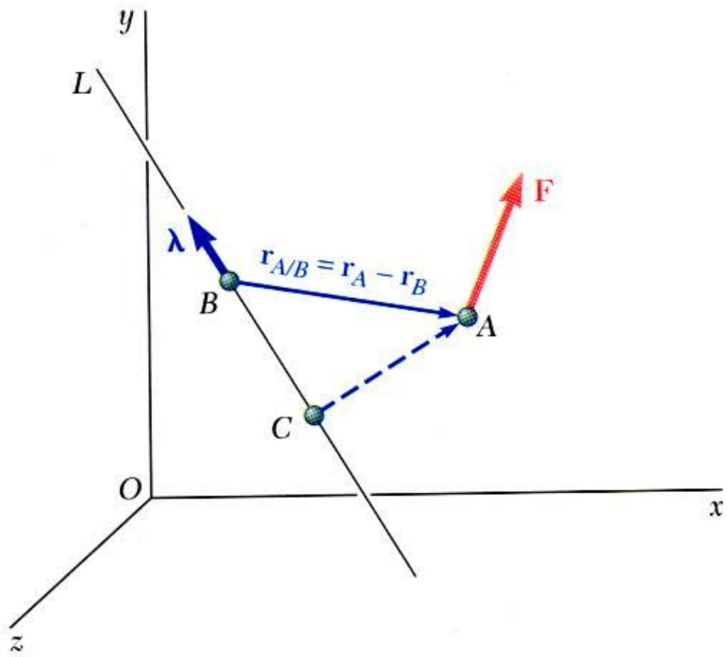
$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$



# Moment of a Force About a Given Axis



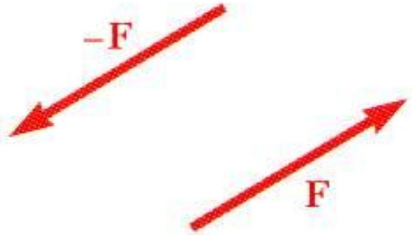

- Moment of a force about an arbitrary axis,

$$M_{BL} = \vec{\lambda} \bullet \vec{M}_B$$
$$= \vec{\lambda} \bullet \left( \vec{r}_{A/B} \times \vec{F} \right)$$
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

- The result is independent of the point *B* along the given axis.

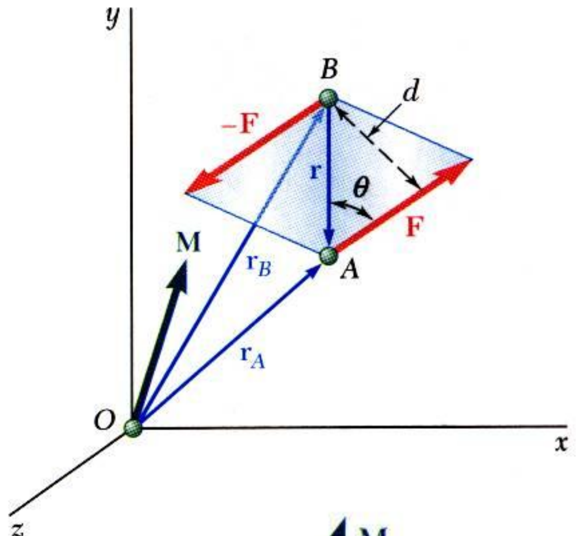
# Moment of a Couple

- Two forces ***F*** and ***-F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

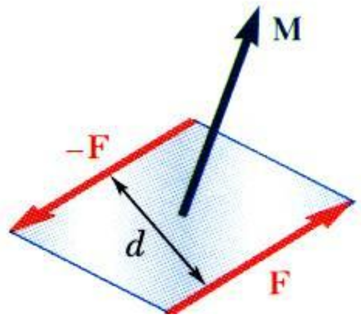



- Moment of the couple,

$$\begin{aligned}\vec{M} &= \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right) \\ &= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F} \\ &= \vec{r} \times \vec{F} \\ M &= rF\sin\theta = Fd\end{aligned}$$



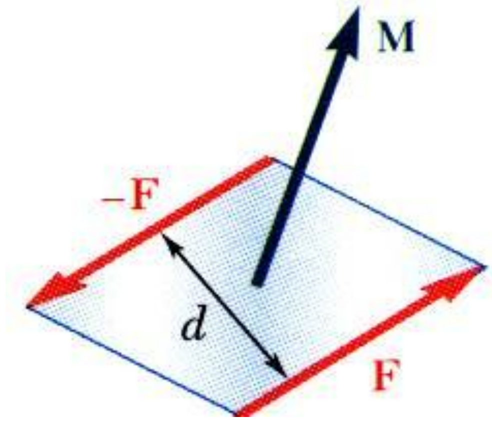

- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

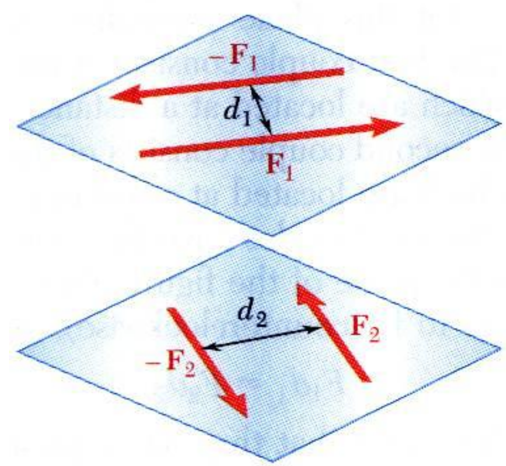


# Moment of a Couple

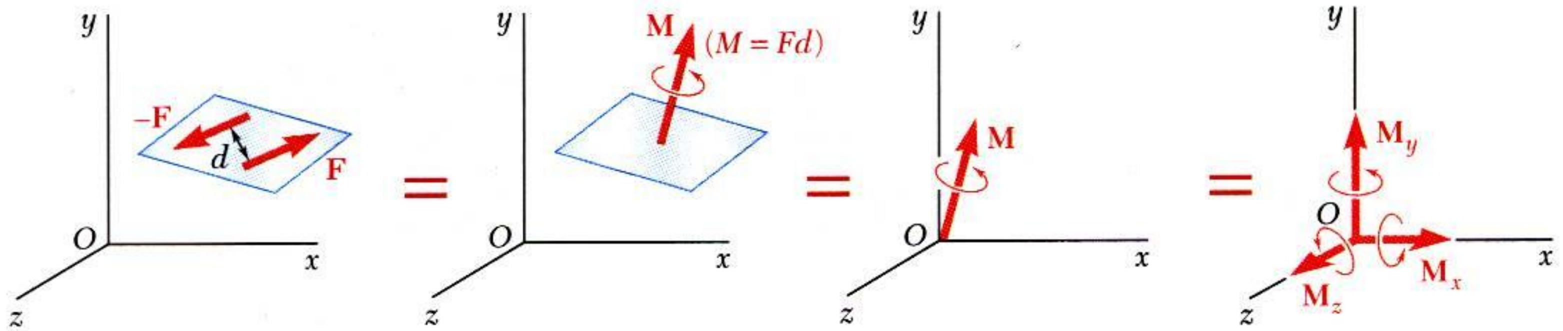
Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



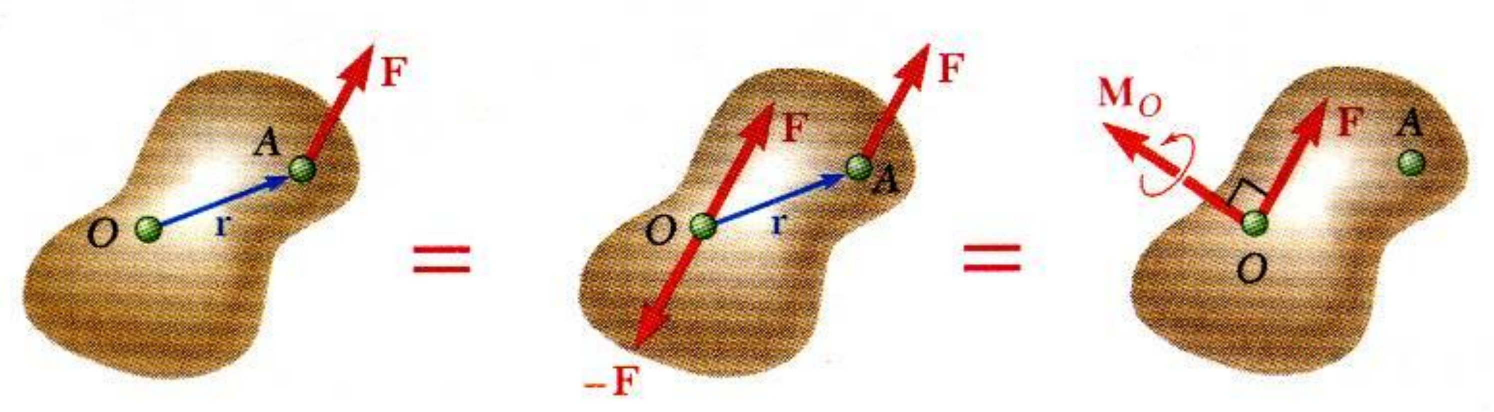



# Couples Can Be Represented by Vectors



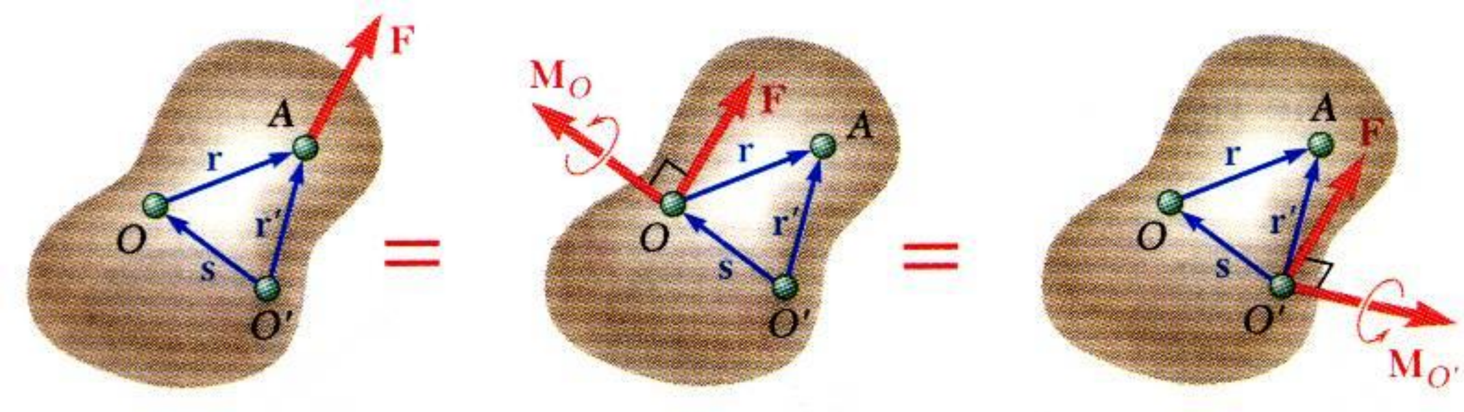

- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.
- *Couple vectors* obey the law of addition of vectors.
- Couple vectors are free vectors, i.e., the point of application is not significant.
- Couple vectors may be resolved into component vectors.

# Resolution of a Force




- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

# Resolution of a Force




- Moving ***F*** from *A* to a different point *O'* requires the addition of a different couple vector $\boldsymbol{M_{O'}}$

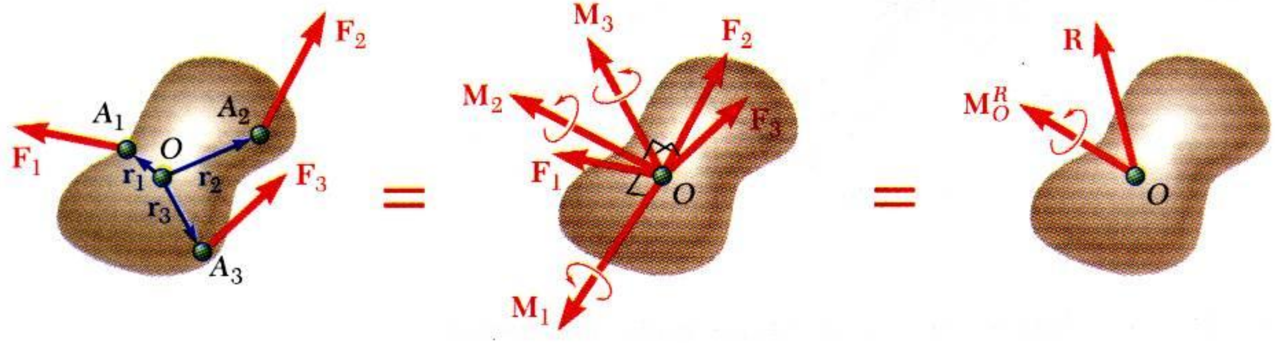
$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F}$$

- The moments of ***F*** about O and *O'* are related,

$$\vec{M}_{O'} = \vec{r}\,' \times \vec{F} = (\vec{r} + \vec{s}) \times \vec{F} = \vec{r} \times \vec{F} + \vec{s} \times \vec{F}$$
$$= \vec{M}_O + \vec{s} \times \vec{F}$$

- Moving the force-couple system from *O* to *O'* requires the addition of the moment of the force at *O* about *O'*.

# Reduction of a System of Forces




- A system of forces may be replaced by a collection of force-couple systems acting a given point *O*

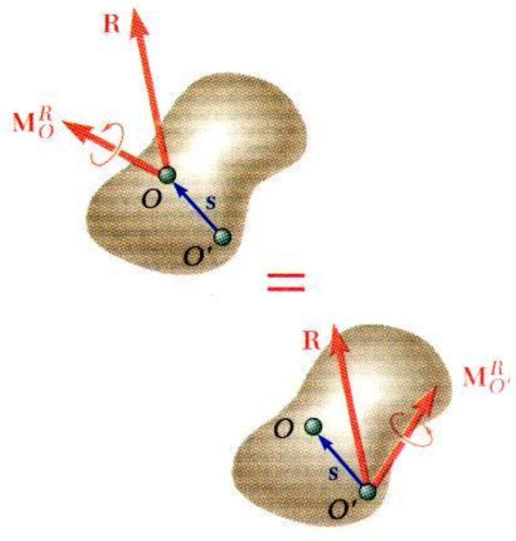
- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left( \vec{r} \times \vec{F} \right)$$

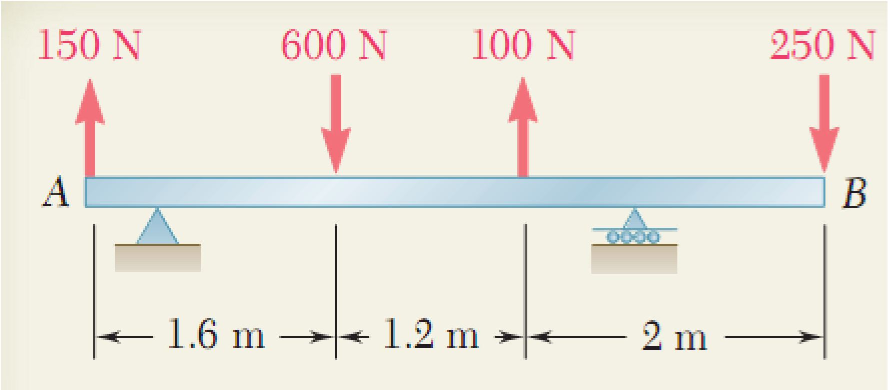
- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

- Two systems of forces are equivalent if they can be reduced to the same force-couple system.



# Sample Problem




For the beam, reduce the system of forces shown to (a) an equivalent force-couple system at *A*, (b) an equivalent force couple system at *B*, and (c) a single force or resultant.

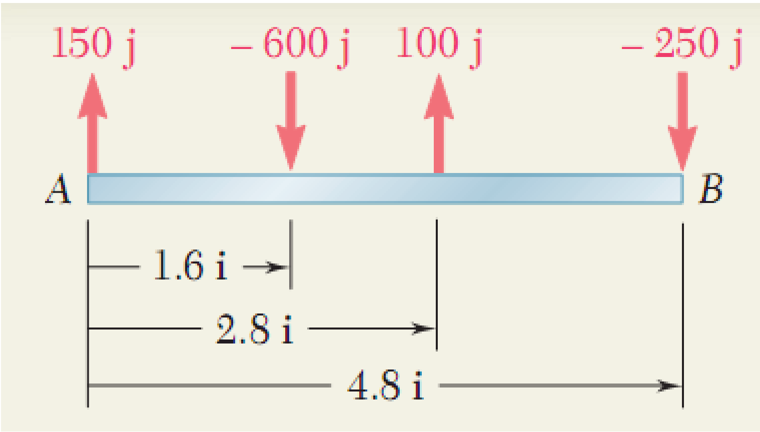
Note: Since the support reactions are not included, the given system will not maintain the beam in equilibrium.

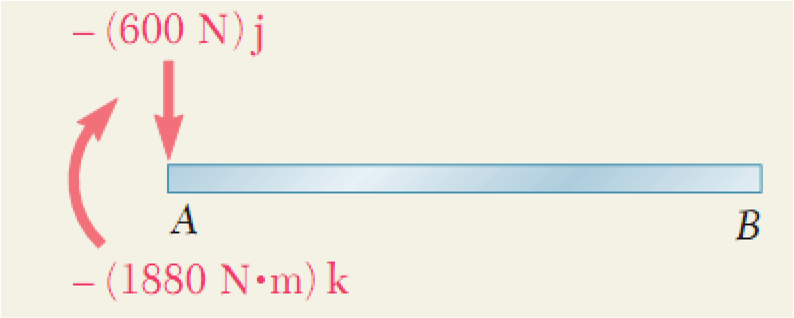
<u>SOLUTION:</u>

a) Compute the resultant force for the forces shown and the resultant couple for the moments of the forces about *A*.

b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

c) Determine the point of application for the resultant force such that its moment about *A* is equal to the resultant couple at *A*.




SOLUTION:

a) Compute the resultant force and the resultant couple at *A*.

$$\vec{R} = \sum \vec{F}$$

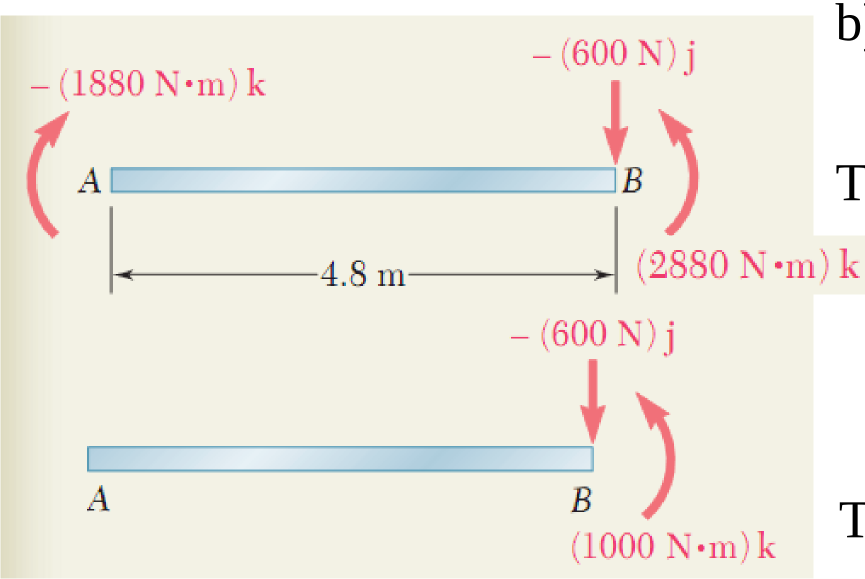
$$= (150\ \text{N})\vec{j} - (600\ \text{N})\vec{j} + (100\ \text{N})\vec{j} - (250\ \text{N})\vec{j}$$

$$\boxed{\vec{R} = -(600\ \text{N})\vec{j}}$$

$$\vec{M}_A^R = \sum \left(\vec{r} \times \vec{F}\right)$$

$$= \left(1.6\vec{i}\right) \times \left(-600\vec{j}\right) + \left(2.8\vec{i}\right) \times \left(100\vec{j}\right) + \left(4.8\vec{i}\right) \times \left(-250\vec{j}\right)$$

$$\boxed{\vec{M}_A^R = -(1880\ \text{N} \cdot \text{m})\vec{k}}$$




b) Find an equivalent force-couple system at *B* based on the force-couple system at *A*.

The force is unchanged by the movement of the force-couple system from *A* to *B*.

$$\vec{R} = -(600 \text{ N})\vec{j}$$

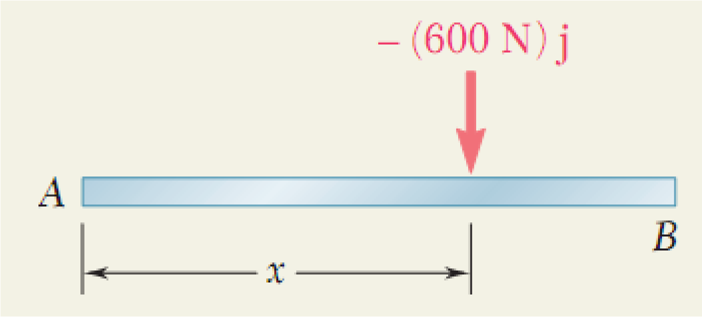
The couple at *B* is equal to the moment about *B* of the force-couple system found at *A*.

$$\vec{M}_B^R = \vec{M}_A^R + \vec{r}_{B/A} \times \vec{R}$$

$$= -(1880 \text{ N} \cdot \text{m})\vec{k} + (-4.8 \text{ m})\vec{i} \times (-600 \text{ N})\vec{j}$$

$$= -(1880 \text{ N} \cdot \text{m})\vec{k} + (2880 \text{ N} \cdot \text{m})\vec{k}$$

$$\vec{M}_B^R = +(1000 \text{ N} \cdot \text{m})\vec{k}$$




c) The resultant of the given system of forces is equal to R, and its point of application must be such that the moment of R about A is equal to $M_A$

$$r \times \vec{R} = \vec{M}_A^R$$

$$(x)\vec{i} \times (-600\text{ N})\vec{j} = -(1880\text{ N}\cdot\text{m})\vec{k}$$

$$-x(600\text{ N})\vec{k} = -(1880\text{ N}\cdot\text{m})\vec{k}$$

$$\vec{R} = -(600\text{ N})\vec{j} \qquad x = 3.13\,m$$